1st-Year Mathematics: Complex Analysis

Problem sheet 4 2017

This problem sheet is a bit different. In the last lecture we introduced a few applications of complex numbers and differential equations. Here you will have the opportunity to take some of the systems we introduced and find solutions using the techniques covered in the course. They are a bit more challenging (i.e. harder than the exam questions), so I expect tutors to pick one of the examples for coverage in the tutorials and for you to try the remainder as homework challenge problems (perhaps your tutors will give you a few hints). For two of the examples you can find more details in my book 'Classical Mechanics From Newton to Einstein' (2nd Ed., Wiley, 2011). This book is referenced as [MWM, p.xx] below.

1. When you do the Electricity and Magnetism course next term, you will encounter the Lorentz force law for a charge q of mass m moving in a magnetic field $\mathbf{B}$:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B}\,. \tag{1}$$

   (a) Show that for a charge moving in the x-y plane under the influence of a magnetic field $\mathbf{B} = B\mathbf{k}$ the equations of motion are

$$m\frac{dv_x}{dt} = qv_yB, \quad m\frac{dv_y}{dt} = -qv_xB\,. \tag{2}$$

   (b) By introducing the complex variable $\tilde{v} = v_x + iv_y$, show that Eqs. (2) may be written as

$$d\tilde{v}/dt = -i\omega_c\tilde{v}\,, \tag{3}$$

   where $\omega_c = \frac{qB}{m}$.

   (c) By solving this equation show that the general solution to Eqs. (2) is

$$\begin{aligned} v_x(t) &= \frac{dx}{dt} = v_x(0)\cos\omega_c t + v_y(0)\sin\omega_c t, \\ v_y(t) &= \frac{dy}{dt} = -v_x(0)\sin\omega_c t + v_y(0)\cos\omega_c t, \end{aligned} \tag{4}$$

   where $v_x(0)$ and $v_y(0)$ are the charge's initial velocity components.

   (d) Integrate these equations again to show that the trajectory of the charge is given by

$$\begin{aligned} x(t) &= x(0) + \omega_c^{-1}\left[v_x(0)\sin\omega_c t + v_y(0)(1-\cos\omega_c t)\right], \\ y(t) &= y(0) + \omega_c^{-1}\left[v_y(0)\sin\omega_c t - v_x(0)(1-\cos\omega_c t)\right]\,, \end{aligned} \tag{5}$$

   where $x(0)$ and $y(0)$ are the coordinates of the charge's initial position.

   (e) Show that the last two equations can be recast as

$$[x - x(0) - v_y(0)/\omega_c]^2 + [y - y(0) + v_x(0)/\omega_c]^2 = v^2/\omega_c^2, \tag{6}$$

   where $v^2 = v_x^2(0) + v_y^2(0)$.

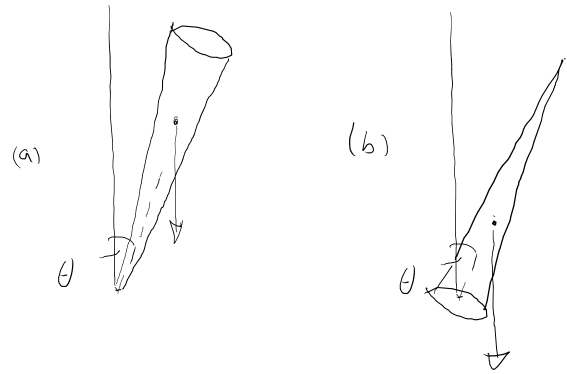

Figure 1: (a) Balancing cue on tip, (b) Balancing cue on base.

(f) Using the above two equations, describe the motion of the charge. What meaning can you assign to a change in sign of ω_0 induced by a change in sign of the charge, q?

2. This question addresses the following: is it easier to balance a snooker cue on its tip (Fig. 1(a)) or on its base (Fig. 1 (b))? The equation we will need is

$$I\frac{d^2\theta}{dt^2} = r\sin\theta Mg \;, \tag{7}$$

where I is called the *moment of inertia*, and r is the distance from the pivot point to the cue's centre of mass. The centre of mass is located at $3L/4$ from the tip of the cue, where L is the cue's length. For (a) $I \approx 3ML^2/5$, while for (b) $I \approx ML^2/10$.

(a) Show that for *small* angles the equation of motion can be written as

$$\frac{d^2\theta}{dt^2} \approx p^2\theta \;, \tag{8}$$

writing down an expression for p for the two cases.

(b) Write down the general solution to Eq. (8) in terms of exponential functions.

(c) Show that if the cue starts at rest with a very small angular deflection $\delta\theta_0$, the solution becomes

$$\theta(t) = \delta\theta_0 \cosh pt \;.$$

(d) On the same axes sketch $\theta(t)$ for the two cases.

(e) Bearing in mind that our solution is only valid for small angles, is is easier to balance a cue on its tip or its base?

3. A Foucault pendulum consists of a mass suspended from a cord of length l suspended vertically (the z direction) at latitude λ, the rotation plane being free to move in the x-y plane (see Figure 2). For small amplitude swings the equations of motion are

$$\ddot{x} - 2\Omega\dot{y} + \omega_0^2 x = 0 \;,$$

$$\ddot{y} + 2\Omega\dot{x} + \omega_0^2 y = 0 \;,$$

where $\omega_0^2 = g/l, \Omega = \omega\sin\lambda$ and ω is the angular rate of the earth's rotation. Note that $\omega_0 \gg \Omega$.

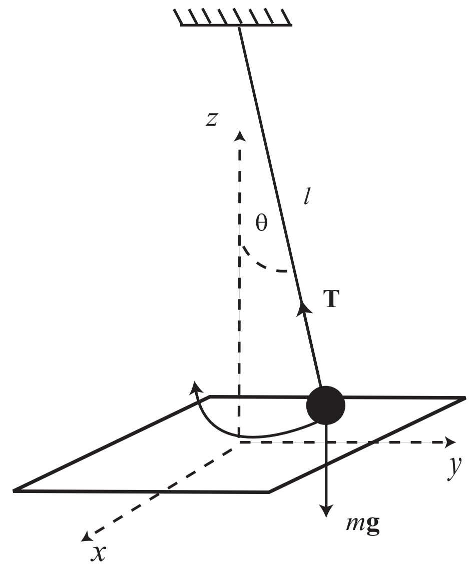

Figure 2: Foucault's pendulum.

(a) By defining a *complex* displacement $z = x + iy$, show that the above equations can be written as the single complex equation

$$\ddot{z} + 2i\Omega\dot{z} + \omega_0^2 z = 0 \ .$$

(b) Show that the roots of the characteristic equation are

$$m = i\left[-\Omega \pm \left(\Omega^2 + \omega_0^2\right)^{1/2}\right] \approx i\left(-\Omega \pm \omega_0\right) \ ,$$

and that the general solution is therefore

$$z(t) \approx \left[z_+ e^{i\omega_0 t} + z_- e^{-i\omega_0 t}\right] e^{-i\Omega t} \ ,$$

where $z_\pm$ are complex constants determined by the initial conditions.

(c) Using the above approximate solution, show that if the pendulum starts from rest $\dot{z}(0) = 0$, with displacement $z(0) = a$ (i.e. real), then (hint: remember $\Omega \ll \omega_0$)

$$z_+ \approx z_- \approx \frac{a}{2} \ .$$

(d) By taking the real and imaginary parts of the complex solution show that

$$x(t) \approx a \cos\omega_0 t \cos\Omega t \ ,$$

$$y(t) \approx -a \cos\omega_0 t \sin\Omega t \ .$$

(e) Use the above solution to make a sketch of the pendulum's motion in the x-y plane. For a more detailed analysis (in which the approximations made here are lifted) - see MWM p.207.

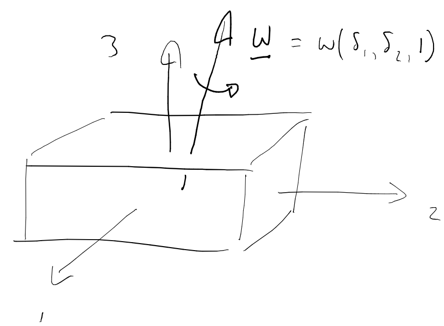

Figure 3: A rotating body has 3 principal axes

4. (MWM, p.211) Every rigid body (e.g. a brick - see Fig. 3) has three *principal axes* to which there are associated three *principal moments of inertia*, $I_{1,2,3}$. In this problem we will assume that all three principal moments are distinct $I_1 \neq I_2 \neq I_3$. When the body spins its *angular velocity* considered as a vector ω, has components $(\omega_1, \omega_2, \omega_3)$ for rotations around the three principal axes respectively. In the absence of any external torque the equations of motion are (known as Euler's equations)

$$I_1 \frac{d\omega_1}{dt} = \omega_2 \omega_3 (I_2 - I_3) \ , \tag{9}$$

$$I_2 \frac{d\omega_2}{dt} = \omega_3 \omega_1 (I_3 - I_1) \ , \tag{10}$$

$$I_3 \frac{d\omega_3}{dt} = \omega_1 \omega_2 (I_1 - I_2) \ , \tag{11}$$

   (a) Imagine the body is rotating mainly about principal axis 3, so that $\omega = \omega(\delta_1, \delta_2, 1)$ where $|\delta_1|, |\delta_2| \ll 1$. Keeping terms to first order in $\delta_{1,2}$, show that the equations of motion reduce to (approximately)

$$I_1 \frac{d\delta_1}{dt} \approx (I_2 - I_3)\, \omega \delta_2 \ , \tag{12}$$

$$I_2 \frac{d\delta_2}{dt} \approx (I_3 - I_1)\, \omega \delta_1 \ . \tag{13}$$

   (b) Hence show that

$$\frac{d^2\delta_1}{dt^2} - q^2 \delta_1 = 0 \ , \tag{14}$$

$$\frac{d^2\delta_2}{dt^2} - q^2 \delta_2 = 0 \ , \tag{15}$$

   where

$$q^2 = \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2} \omega^2 \ . \tag{16}$$

   (c) Using what you know about the nature of solutions to equations (14) and (15) for different signs of q^2 deduce that the motion is *stable* (i.e. $\delta_{1,2}$ do not grow exponentially) if either $I_3 < I_{1,2}$, or $I_3 > I_{1,2}$, and is *unstable* (i.e. $\delta_{1,2}$ do grow exponentially) if I_3 lies between the other two principal moments, i.e. $I_1 < I_3 < I_2$ say.

   (d) Try it with a matchbox!