1st-Year Mathematics: Complex Analysis

Problem sheet 4 - Solutions 2017

1.
$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B}\,. \tag{1}$$

(a) For a charge moving in the x-y plane under the influence of a magnetic field $\mathbf{B} = B\mathbf{k}$ we have
$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & 0 \\ 0 & 0 & B \end{vmatrix} = v_yB\mathbf{i} - v_xB\mathbf{j}\,.$$
The x and y components of Eqs. (1) are therefore
$$m\frac{dv_x}{dt} = qv_yB, \quad m\frac{dv_y}{dt} = -qv_xB\,. \tag{2}$$

(b) Setting $\tilde{v} = v_x + iv_y$, we have
$$d\tilde{v}/dt = \frac{dv_x}{dt} + i\frac{dv_y}{dt} = \omega_c v_y - i\omega_c v_x = -i\omega_c\tilde{v}\,, \tag{3}$$
where $\omega_c = \frac{qB}{m}$.

(c) The solution to the above equation is
$$\tilde{v}(t) = \tilde{v}(0)e^{-i\omega_c t}\,,$$
or, after setting $\tilde{v}(0) = v_x(0) + iv_y(0)$, using Euler's identity, and multiplying out
$$v_x(t) = \frac{dy}{dt} = v_x(0)\cos\omega_c t + v_y(0)\sin\omega_c t,$$
$$v_y(t) = \frac{dy}{dt} = -v_x(0)\sin\omega_c t + v_y(0)\cos\omega_c t\,. \tag{4}$$

(d) Integrating the first of these equations
$$\begin{aligned} x(t) &= x(0) + \int_0^t v_x(t')dt' = x(0) + \int_0^t \left[v_x(0)\cos\omega_c t' + v_y(0)\sin\omega_c t'\right]dt' \\ &= x(0) + \omega_c^{-1}\left[v_x(0)\sin\omega_c t' - v_y(0)\cos\omega_c t'\right]_0^t \\ &= x(0) + \omega_c^{-1}\left[v_x(0)\sin\omega_c t + v_y(0)\left(1-\cos\omega_c t\right)\right]\,, \end{aligned} \tag{5}$$
where $x(0)$ is charge's initial x-coordinate. The corresponding equation for $y(t)$ is derived similarly as
$$y(t) = y(0) + \omega_c^{-1}\left[v_y(0)\sin\omega_c t - v_x(0)(1-\cos\omega_c t)\right]\,.$$

(e) The last two equations can be combined as
$$[x - x(0) - v_y(0)/\omega_c]^2 + [y - y(0) + v_x(0)/\omega_c]^2 = v^2/\omega_c^2 \tag{6}$$
where $v^2 = v_x^2(0) + v_y^2(0)$.

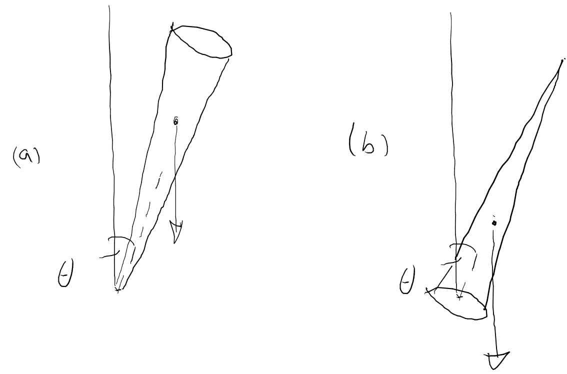

Figure 1: (a) Balancing cue on tip, (b) Balancing cue on base.

(f) The last equation above shows that the charge describes a circle of radius v/ω_c centred at $(x,y) = (x(0)+v_y(0)/\omega_c, y(0)-v_x(0)/\omega_c)$. Changing the sign of ω_0 by changing the sign of q causes the sense of rotation to be reversed.

2. (a) For small angles $\sin\theta \approx \theta$, so the given equation of motion becomes

$$\frac{d^2\theta}{dt^2} = p^2\theta \ ,$$

where $p^2 = Mgr/I$

(b) For balancing on the tip we have $I \approx 3ML^2/5$ and $r = 3L/4$ so

$$p_{\text{tip}} = \left(\frac{Mgr}{I}\right)^{1/2} = \left(\frac{Mg \times 3L/4}{3ML^2/5}\right)^{1/2} = \left(\frac{5g}{4L}\right)^{1/2} \ .$$

For balancing on the base we have $I \approx ML^2/10$ and $r = L/4$ so

$$p_{\text{base}} = \left(\frac{Mgr}{I}\right)^{1/2} = \left(\frac{Mg \times L/4}{ML^2/10}\right)^{1/2} = \left(\frac{5g}{2L}\right)^{1/2} \ .$$

So $p_{\text{base}} > p_{\text{tip}}$.

(c) Using a trial solution we establish that $e^{\pm pt}$ are solutions. The general solution is a linear combination of these:

$$\theta(t) = Ae^{pt} + Be^{-pt} \ ,$$

(d) If $\theta(0) = \delta\theta_0$, then

$$\delta\theta_0 = A + B \ .$$

If also $\left.\frac{d\theta}{dt}\right|_{t=0} = 0$ then

$$0 = p(A-B) \ .$$

From these equations we deduce that $A = B = \delta\theta_0/2$. The general solution then becomes

$$\theta(t) = \frac{\delta\theta_0}{2}\left(e^{pt} + e^{-pt}\right) = \delta\theta_0 \cosh pt \ .$$

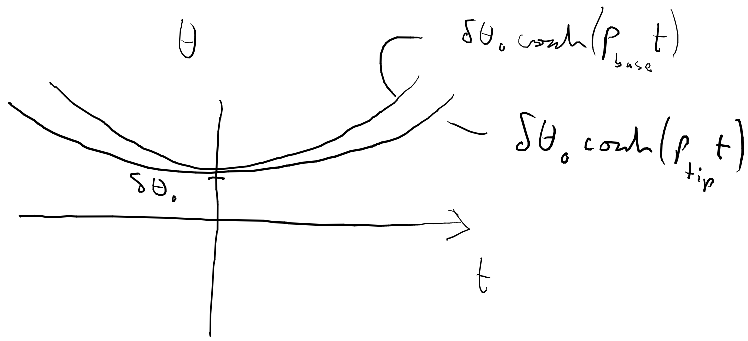

Figure 2: Sketch of $\theta(t)$ for the two cases.

(e) See sketch below.

(f) Since $p_{\rm base} > p_{\rm tip}$ it's easier to balance the cue on its tip.

3. (a) Taking the two equations

$$\ddot{x} - 2\Omega\dot{y} + \omega_0^2 x = 0 \; ,$$
$$\ddot{y} + 2\Omega\dot{x} + \omega_0^2 y = 0 \; ,$$

and adding the first to i times the second yields

$$(\ddot{x} + i\ddot{y}) - 2\Omega(\dot{y} - i\dot{x}) + \omega_0^2(x + iy) = 0$$

or

$$(\ddot{x} + i\ddot{y}) + 2i\Omega(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) = 0 \; ,$$

or

$$\ddot{z} + 2i\Omega\dot{z} + \omega_0^2 z = 0 \; .$$

(b) Substituting e^{mt} into the complex equation of motion yields the characteristic equation

$$m^2 + 2i\Omega m + \omega_0^2 = 0 \; .$$

Solving the quadratic yields

$$m = -i\Omega \pm i(\omega_0^2 + \Omega^2)^{1/2} \approx -i\Omega \pm i\omega_0 \; ,$$

since $\omega_0 \gg \Omega$. The general solution is a linear combination of e^{mt} and e^{-mt}, i.e.

$$z(t) \approx \left[z_+ e^{i\omega_0 t} + z_- e^{-i\omega_0 t}\right] e^{-i\Omega t} \; ,$$

where $z_\pm$ are complex constants determined by the initial conditions.

(c) Setting $z(0) = a$ and $\dot{z}(0) = 0$, means that

$$a = z_+ + z_- \; , \;\; \text{and} \;\; 0 = (\omega_0 - \Omega)z_+ - (\omega_0 + \Omega)z_- \approx \omega_0 z_+ - \omega_0 z_- \; .$$

Solving these simultaneously yields

$$z_+ = z_1 = \frac{a}{2} \; .$$

(d) The solution now reads

$$z(t) \approx \frac{a}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right] e^{-i\Omega t} \ .$$

Hence

$$x \approx \mathrm{Re}(z) = \frac{a}{2}\{\cos[(\omega_0 - \Omega)t] + \cos[(\omega_0 + \Omega)t]\} = a\cos\omega_0 t\cos\Omega t \ ,$$
$$y \approx \mathrm{Im}(z) = \frac{a}{2}\{\sin[(\omega_0 - \Omega)t] - \sin[(\omega_0 + \Omega)t]\} = -a\cos\omega_0 t\sin\Omega t \ .$$

(e) See figure below.

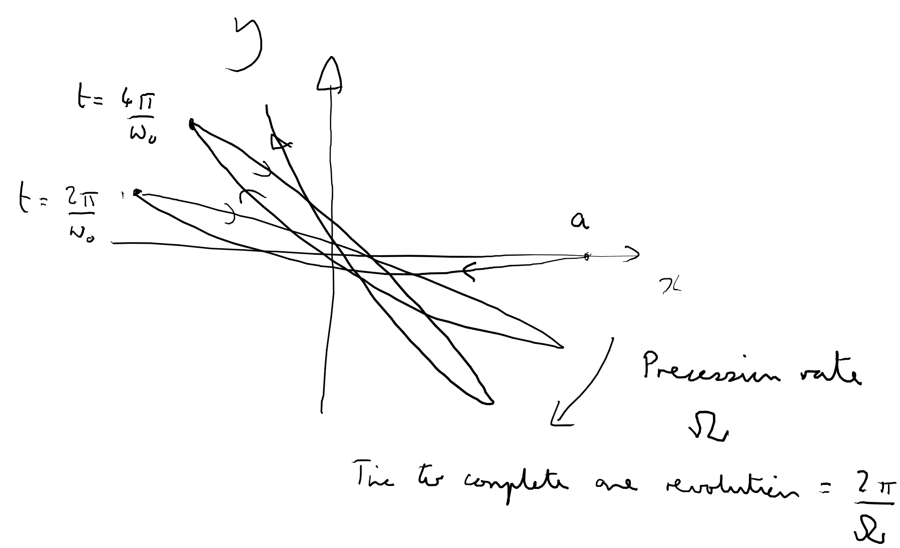

Figure 3: Trajectory of Foucault pendulum in *x*-*y* plane.

4. (a) The rhs of the third Euler equation is $\propto \omega_1\omega_2 = \delta_1\delta_2 \approx 0$. Then $\omega_3 = \omega$ is approximately constant, and that the remaining two equations then become

$$I_1\frac{d\delta_1}{dt} = (I_2 - I_3)\,\omega\delta_2 \ , \quad (7)$$
$$I_2\frac{d\delta_2}{dt} = (I_3 - I_1)\,\omega\delta_1 \ . \quad (8)$$

(b) By differentiating the first equation and using the second

$$\begin{aligned}\frac{d^2\delta_1}{dt^2} &= \frac{(I_2 - I_3)}{I_1}\omega\frac{d\delta_2}{dt} \\ &= \frac{(I_2 - I_3)}{I_1}\omega\frac{(I_3 - I_1)}{I_2}\omega\delta_1 \\ &= \frac{(I_3 - I_1)\,(I_2 - I_3)}{I_1 I_2}\omega^2 = q^2\delta_1 \ ,\end{aligned}$$

where

$$q^2 = \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2} \omega^2 \ . \tag{9}$$

The second equation is derived similarly.

(c) If $q^2 < 0$ then the general solutions for $\delta_{1,2}$ are of the form

$$\delta_{1,2} = Ae^{iqt} + Be^{-iqt} \ ,$$

indicating oscillatory motion of the angular perturbations $\delta_{1,2}$. These oscillations do not grow and can only diminish through friction. The tumbling is *stable*. The condition $q^2 < 0$ occurs when either $I_3 > I_{1,2}$ or $I_3 < I_{1,2}$.

If $q^2 > 0$ then the general solutions for $\delta_{1,2}$ are of the form

$$\delta_{1,2} = Ae^{qt} + Be^{-qt} \ ,$$

implying exponential growth (at least until $|\delta_{1,2}| \ll 1$ breaks down). This indicates an *unstable* tumbling of the body. The condition $q^2 > 0$ occurs when either $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$, i.e. tumbling about the intermediate axis is unstable.