1st-Year Mathematics: Complex Analysis

Problem sheet 3 2017

For tutorials

We will in this problem consider coupled differential equations. First from a purely mathematical point of view and subsequently going on to a physics example.
Consider two systems labelled **A** and **B** which are weakly coupled according to the ODE

$$\frac{dx_A}{dt} = x_A + \varepsilon x_B$$
$$\frac{dx_B}{dt} = \varepsilon x_A + x_B \,,$$

where $\varepsilon \ll 1$.

1. Show that if we define two new variables

$$x_1 = x_A + x_B$$
$$x_2 = x_A - x_B$$

   then the ODE expressed in terms of x_1 and x_2 are not coupled.

2. Solve the two separate ODEs in x_1 and x_2 and use this to find the general solutions for x_A and x_B.

3. If $x_A = x_B = 1$ at $t = 0$, show that the solutions are of the form

$$x_A = x_B = e^{(1+\varepsilon)t} \,.$$

   Discuss if this makes sense in terms of the limit $\varepsilon \to 0$.

4. Compare the solution for x_A (with $x_A = x_B = 1$ at $t = 0$) to the one with no coupling ($\varepsilon = 0$) (maybe form the ratio between the two) and discuss why it can be very difficult to measure if a system has a small coupling.

5. Determine the type of the differential equation

$$\frac{dy}{dx} + y = e^{-x} \,.$$

   Solve it using an integrating factor for the initial condition $y(0) = -1$.

Homework

1. **Newton's law of cooling** states that the temperature of a body changes at a rate proportional to the difference in temperature between the body and that of its environment. The differential equation for the temperature $T(t)$ of a body at time t in an environment whose ambient temperature is θ is

$$\frac{dT}{dt} = -k(T-\theta)\,,$$

   where k is a positive constant. Show that the solution of this equation with the initial condition $T(0) = T_0$ is

$$T(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,.$$

   Hint: Begin by obtaining the differential equation for $u(t) = T(t) - \theta$, using the fact that θ is a constant. Note that the corresponding initial condition is $u(0) = T_0 - \theta$.

2. Consider the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,,$$

   with the initial conditions

$$x(0) = x_0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = x_0'\,.$$

   Show that the solution to this initial value problem is

$$x(t) = x_0 \cos(\omega_0 t) + \frac{x_0'}{\omega_0}\sin(\omega_0 t)\,.$$

3. According to the theory of beam bending, the deflection y of a beam from the horizontal is determined by a fourth-order differential equation of the form

$$\frac{d^4y}{dx^4} - y = 0\,.$$

   Determine the general solution of this equation (hint: use a trial solution). How many auxiliary conditions are required to determine the arbitrary constants in the general solution?