1st-Year Mathematics: Complex Analysis

Problem sheet 3 - Solutions 2017

For tutorials

1. With the definitions of x_1 and x_2 we have

$$\frac{dx_1}{dt} = \frac{dx_A}{dt} + \frac{dx_B}{dt}$$
$$\frac{dx_2}{dt} = \frac{dx_A}{dt} - \frac{dx_B}{dt} \ .$$

Inserting into the ODE and rearranging then yields

$$\frac{dx_1}{dt} = (1+\varepsilon)x_1$$
$$\frac{dx_2}{dt} = (1-\varepsilon)x_2$$

and we see that this is just two independent ODEs.

2. Simple separation of variables can be used to solve these ODEs (or we can just write down the known exponential solutions). We have

$$x_1 = C_1 e^{(1+\varepsilon)t}$$
$$x_2 = C_2 e^{(1-\varepsilon)t} \ ,$$

where C_1 and C_2 are independent integration constants. Transforming this back into x_A and x_B gives

$$x_A = C_1 e^{(1+\varepsilon)t} + C_2 e^{(1-\varepsilon)t}$$
$$x_B = C_1 e^{(1+\varepsilon)t} - C_2 e^{(1-\varepsilon)t} \ .$$

3. Inserting the boundary conditions and solving for C_1 and C_2 gives $C_1 = 1$ and $C_2 = 0$. Thus the solutions are

$$x_A = e^{(1+\varepsilon)t}$$
$$x_B = e^{(1+\varepsilon)t} \ .$$

In the limit of $\varepsilon \to 0$ the original ODEs turn uncoupled with solutions

$$x_A = e^t$$
$$x_B = e^t \ ,$$

which is exactly what happens in the limit of the solutions above. This seems very reasonable.

4. The ratio between the coupled and the uncoupled solution only differ by the factor $\exp(\varepsilon t)$. If ε is very small it will take a long time for the coupled system to have any noticeable difference to the uncoupled one. Over such long time periods the calibration of the measurement system may drift and it might not be possible at all to find the effect.

5. We see that this is a first order ordinary differential equation. It is linear (first degree) and inhomogeneous.

   We solve it using the method of an integrating factor. We have

$$P(x) = 1 \qquad Q(x) = e^{-x},$$

   which then gives

$$I = \int P(x)dx = x$$

   and thus

$$e^I = e^x \qquad e^{-I} = e^{-x}.$$

   The solution is then

$$y = e^{-x} \int e^x e^{-x} dx + Ce^{-x} = xe^{-x} + Ce^{-x},$$

   where C is the integration constant. By inserting $x = 0$, we see that $C = -1$ for the required initial condition. Thus the solution is

$$y = (x-1)e^{-x},$$

   which is easily tested by evaluating $y(0)$ and by inserting into the original ODE.

Homework

1. Beginning with the differential equation

$$\frac{dT}{dt} = -k(T - \theta), \tag{1}$$

   we introduce the function

$$u(t) = T(t) - \theta.$$

   Since θ is a constant, the differential equation for u is

$$\frac{du}{dt} = \frac{dT}{dt} = -k(T - \theta) = -ku,$$

   with the initial condition

$$u(0) = T(0) - \theta = T_0 - \theta.$$

   As discussed in the lectures, the solution to this equation is

$$u(t) = A\,\mathrm{e}^{-kt},$$

   where A is a constant determined by the initial condition:

$$u(0) = T_0 - \theta = A.$$

In terms of the original function T, this solution is

$$T(t) = \theta + u(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,. \tag{2}$$

Notice that (1) can also be integrated directly. By separating the variables in the equation,

$$\frac{dT}{T-\theta} = -k dt\,,$$

and integrating,

$$\int_{T_0}^{T(t)} \frac{dT}{T-\theta} = -k\int_0^t ds\,,$$

we obtain

$$\ln(T-\theta)\Big|_{T_0}^{T(t)} = \ln\left[\frac{T(t)-\theta}{T_0-\theta}\right] = -kt\,.$$

Solving for $T(t)$,

$$T(t) = \theta + (T_0 - \theta)\,\mathrm{e}^{-kt}\,,$$

which is the same as (2).

2. We solve the equation of motion of a classical undamped harmonic oscillator with natural frequency ω_0,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0\,, \tag{3}$$

with a trial solution $x(t) = e^{mt}$. Substituting this expression into the equation yields

$$m^2\, e^{mt} + \omega_0^2\, e^{mt} = (m^2 + \omega_0^2)\, e^{mt} = 0\,.$$

The characteristic equation is

$$m^2 + \omega_0^2 = (m - i\omega_0)(m + i\omega_0) = 0\,,$$

which has roots $m_1 = -i\omega_0$ and $m_2 = i\omega_0$. The general solution to (3) is

$$x(t) = A\, e^{-i\omega_0 t} + B\, e^{i\omega_0 t}\,, \tag{4}$$

where A and B are determined by the initial conditions,

$$x(0) = x_0\,, \qquad \frac{dx}{dt}\Big|_{t=0} = x_0'\,,$$

which correspond to an initial displacement x_0 and an initial velocity x_0'. Substitution of (4) into the initial conditions produces

$$\begin{aligned} x(0) &= A + B = x_0\,, \\ \frac{dx}{dt}\Big|_{t=0} &= -i\omega_0 A + i\omega_0 B = x_0'\,. \end{aligned}$$

After dividing both sides of the second equation by ω_0 and multiplying both sides by i, we obtain the two simultaneous equations for A and B in the form:

$$A + B = x_0\,,$$
$$A - B = \frac{ix_0'}{\omega_0}\,.$$

These equations are easily solved to obtain

$$A = \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right)\,,$$
$$B = \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right)\,.$$

Thus, the solution to the initial-value problem is

$$\begin{aligned}
x(t) &= \frac{1}{2}\left(x_0 + \frac{ix_0'}{\omega_0}\right) e^{-i\omega_0 t} + \frac{1}{2}\left(x_0 - \frac{ix_0'}{\omega_0}\right) e^{i\omega_0 t} \\
&= x_0\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) - \frac{ix_0'}{\omega_0}\left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2}\right) \\
&= x_0\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) + \frac{x_0'}{\omega_0}\left(\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}\right) \\
&= x_0 \cos\omega_0 t + \frac{x_0'}{\omega_0}\sin\omega_0 t\,.
\end{aligned}$$

3. Since this is a differential equation with constant coefficients, we attempt to find solutions with a trial solution of the form $y(x) = e^{mx}$. Substituting this expression into the differential equation yields

$$m^4\,e^{mx} - e^{mx} = (m^4 - 1)\,e^{mx} = 0\,.$$

The characteristic equation is identified as

$$m^4 - 1 = 0\,,$$

which can be factored as

$$m^4 - 1 = (m^2 - 1)(m^2 + 1) = (m - 1)(m + 1)(m - i)(m + i) = 0\,,$$

which yields four distinct roots: $m = -1, 1, -i, i$. Accordingly, there are four solutions of the differential equation:

$$y_1(x) = e^{-x}\,, \quad y_2(x) = e^{x}\,, \quad y_3(x) = e^{-ix}\,, \quad y_4(x) = e^{ix}\,.$$

The general solution is a general linear combination of these solutions,

$$y(x) = A\,e^{-x} + B\,e^{x} + C\,e^{-ix} + D\,e^{ix}\,,$$

where *four* initial conditions are required to determine the four constants A, B, C, and D.