1st-Year Mathematics: Complex Analysis

Problem Sheet 2 2017

For tutorials

1. The n**th roots of unity** are the solutions to the equation $z^n = 1$ for any positive integer n. Use the polar representation of complex numbers, $z = e^{i\theta}$, to obtain the roots

$$\omega_k = \exp\left(\frac{2\pi i k}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right),$$

for $k = 0, 1, 2, \ldots, n-1$.

2. (a) Use the polar representation of complex numbers to show that $(z^n)^* = (z^*)^n$ for any complex number z and positive integer n.

   (b) Use the result in (a) to show that complex roots of unity occur in complex conjugate pairs.

   (c) How many real and how many complex nth roots of unity are there if (i) n is even and (ii) if n is odd.

3. Find the sum of the nth roots of unity,

$$\sum_{k=0}^{n-1} \omega_k = \omega_0 + \omega_1 + \cdots + \omega_{n-1} \quad \text{where } n > 1 \ .$$

You may solve this either geometrically or by using that

$$\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots x^{n-1} = \frac{1-x^n}{1-x} \, .$$

Homework

1. Consider the nth roots of a complex number $w = \rho\, e^{i\phi}$: $z^n = w$. In lectures, we derived the following formula for the roots of this equation:

$$z_k = \rho^{1/n} \exp\left[i\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)\right], \tag{1}$$

for $k = 0, 1, 2, \ldots, n-1$. Show how z_k can be expressed in terms of the nth roots of unity. What is the sum $z_0 + z_1 + \cdots + z_{n-1}$?

2. We have seen that there are similarities and differences between e^x and e^z. Here follows some additional examples. In each case either prove your answer or provide an example where the statement is false.

(a) The function e^x is increasing, i.e. $e^{x_1} < e^{x_2}$ if $x_1 < x_2$. If $|z_1| < |z_2|$, is $|e^{z_1}| < |e^{z_2}|$?

(b) The function e^x never vanishes. Can e^z vanish?

(c) $e^x = 1$ if and only if $x = 0$. Do we have that $e^z = 1$ if and only if $z = 0$?

3. Express the following functions of $z = x + iy$ in the form of $u(x,y) + i\,v(x,y)$, where u and v are the real and imaginary parts of these functions.

   (a) $\sin(2z)$ (b) $\cos(z^2)$ (c) $2z + \sin z$ (d) $z\cos z$

4. Explain why the magnitudes of $\cos z$ and $\sin z$ are not bounded.

5. Find the principal values of the following logarithms

   (a) $\ln(2\,i)$ (b) $\ln(-3-3\,i)$ (c) $\ln(4\,e^{\frac{1}{4}i\pi})$