1st-Year Mathematics: Complex Analysis

Solutions to Problem Sheet 2 2017

Tutorial

1. Using the polar representation $z = r\,e^{i\theta}$, the nth roots unity are defined by the equation

$$z^n = \left(r\,e^{i\,\theta}\right)^n = r^n\,e^{in\theta} = 1\,. \tag{1}$$

This yields $r^n = 1$, whose positive solution is $r = 1$, and

$$n\theta = 0,\ 2\pi,\ 4\pi,\ \ldots,\ 2\pi(n-1)\,, \tag{2}$$

since there must be n roots, or,

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2\pi(n-1)}{n}\,. \tag{3}$$

Thus, the nth roots of unity, which are signified by ω_k, are

$$\omega_k = \exp\left(\frac{2\pi i\,k}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i\,\sin\left(\frac{2\pi k}{n}\right), \tag{4}$$

for $k = 0, 1, 2, \ldots, n-1$.

2. (a) Using the polar representation $z = r\,e^{i\theta}$,

$$(z^n)^* = \left[\left(r\,e^{i\theta}\right)^n\right]^* = \left(r^n\,e^{in\theta}\right)^* = r^n\,e^{-in\theta} = \left(r\,e^{-i\theta}\right)^n = (z^*)^n\,. \tag{5}$$

(b) The nth roots of unity are solutions of $z^n = 1$. Taking the complex conjugate of this equation, and using the result of (a), yields $(z^*)^n = 1$. Thus, if z is an nth root of unity, then so is z^*. Thus, complex roots of unity occur in complex conjugate pairs.

(c) If n is even, then -1 and 1 are both nth roots of unity. These are the only real solutions, so the remaining $n-2$ solutions must be complex conjugate pairs. If n is odd, the only real solution is 1, so the remaining $n-1$ solutions are complex conjugate pairs.

3. Consider the sum

$$S_{n-1} = 1 + x + x^2 + \cdots + x^{n-1} = \sum_{k=0}^{n-1} x^n\,. \tag{6}$$

By multiplying both sides of this equation by x, we obtain

$$xS_{n-1} = x + x^2 + x^3 + \cdots + x^n\,. \tag{7}$$

Taking the difference,

$$S_{n-1} - xS_{n-1} = S_n(1-x) = 1 - x^n\,, \tag{8}$$

and solving for S_n, yields

$$S_n = \frac{1-x^n}{1-x}\,. \tag{9}$$

Consider now the sum of the nth roots of unity. Using the exponential representation in (4), we have

$$\begin{aligned}\omega_0+\omega_1\,+\,\omega_2+\cdots+\omega_{n-1} &= 1+\exp\left(\frac{2\pi i}{n}\right)+\exp\left(\frac{4\pi i}{n}\right)+\cdots+\exp\left[\frac{2\pi i(n-1)}{n}\right]\\ &= 1+\exp\left(\frac{2\pi i}{n}\right)+\left[\exp\left(\frac{2\pi i}{n}\right)\right]^2+\cdots+\left[\exp\left(\frac{2\pi i}{n}\right)\right]^{n-1}, \end{aligned} \tag{10}$$

which has the same form as the series in (6) with $x=e^{2\pi i/n}$. Hence, the sum of these roots is given by (9):

$$S_n = \frac{1-e^{2\pi i}}{1-e^{2\pi i/n}}\,. \tag{11}$$

Since $e^{2\pi i}=1$ and $e^{2\pi i/n}\neq 1$ (for $n>1$), $S_n=0$, so

$$\omega_0+\omega_1+\omega_2+\cdots+\omega_{n-1}=0\,. \tag{12}$$

Homework

1. Using the basic properties of the exponential, we have

$$z_k=\rho^{1/n}\exp\left[i\left(\frac{\phi}{n}+\frac{2k\pi}{n}\right)\right]=\rho^{1/n}\exp\left(\frac{i\phi}{n}\right)\exp\left(\frac{2\pi i k}{n}\right)=\rho^{1/n}\exp\left(\frac{i\phi}{n}\right)\omega_k\,, \tag{13}$$

where the ω_k are the nth roots of unity in (4). For notational convenience (this is not an essential step), we can define the **principal root** $w_p^{1/n}$ by

$$w_p^{1/n}\equiv\rho^{1/n}\exp\left(\frac{i\phi}{n}\right), \tag{14}$$

in which case $z_k=w_p^{1/n}\omega_k$. The sum of the z_k is

$$\begin{aligned} z_0+z_1+z_2+\cdots+z_{n-1} &= w_p^{1/n}\omega_0+w_p^{1/n}\omega_1+w_p^{1/n}\omega_2+\cdots+w_p^{1/n}\omega_{n-1}\\ &= w_p^{1/n}\big(\omega_0+\omega_1+\omega_2+\cdots+\omega_{n-1}\big)\\ &= 0\,, \end{aligned} \tag{15}$$

where we have invoked (12).

2. (a) We have that

$$\left|e^z\right|=\left|e^{x+iy}\right|=\left|e^x\right|\left|e^{iy}\right|=e^x\,,$$

because e^x is a positive real number. Since $|z|=(x^2+y^2)^{1/2}$, and the magnitude of e^z depends only on the real part of z, there is not a monotonic relationship between $|z|$ and $|e^z|$. A counter example is $z_1=1$ and $z_2=2i$, where we have $|z_1|=1$, $|z_2|=2$ and $|e^{z_1}|=e$, $|e^{z_2}|=1$.

(b) Given that
$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)\,,$$
we see that there is no value of y for which both $\sin y$ and $\cos y$ vanish.

(c) As in (b),
$$e^z = e^x(\cos y + i\sin y)\,.$$
Thus, $e^z = 1$ if $x = 0$ and if $y = 0, 2\pi, 4\pi \ldots$. Only if we restrict the range of y to $0 \le y < 2\pi$ is this statement true.

3. We use the following decompositions for this problem:

$$\begin{aligned}
\sin(a+ib) &= \frac{1}{2i}\big(e^{i(a+ib)} - e^{-i(a+ib)}\big) \\
&= \frac{1}{2i}\big(e^{-b}e^{ia} - e^{b}e^{-ia}\big) \\
&= \frac{1}{2i}\big[e^{-b}(\cos a + i\sin a) - e^{b}(\cos a - i\sin a)\big] \\
&= \frac{1}{2i}\big[\big(e^{-b} - e^{b}\big)\cos a + i\big(e^{-b} + e^{b}\big)\sin a\big] \\
&= \sin a\cosh b + i\cos a\sinh b\,. \qquad (16)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos(a+ib) &= \frac{1}{2}\big(e^{i(a+ib)} + e^{-i(a+ib)}\big) \\
&= \frac{1}{2}\big(e^{-b}e^{ia} + e^{b}e^{-ia}\big) \\
&= \frac{1}{2}\big[e^{-b}(\cos a + i\sin a) + e^{b}(\cos a - i\sin a)\big] \\
&= \frac{1}{2}\big[\big(e^{-b} + e^{b}\big)\cos a + i\big(e^{-b} - e^{b}\big)\sin a\big] \\
&= \cos a\cosh b - i\sin a\sinh b\,, \qquad (17)
\end{aligned}$$

(a) We use (16) with $2z = 2x + 2iy$, so $a = 2x$ and $b = 2y$:

$$u(x,y) = \mathrm{Re}(\sin 2z) = \sin 2x\cosh 2y\,,$$

$$v(x,y) = \mathrm{Im}(\sin 2z) = \cos 2x\sinh 2y\,.$$

(b) We use (17) with $z^2 = x^2 - y^2 + 2ixy$, so $a = x^2 - y^2$ and $b = 2xy$:

$$u(x,y) = \mathrm{Re}(\cos z^2) \;=\; \cos(x^2 - y^2)\cosh(2xy)\,,$$

$$v(x,y) = \mathrm{Im}(\cos z^2) \;=\; -\sin(x^2 - y^2)\sinh(2xy)\,.$$

(c) We use (16) with $z = x + iy$, so $a = x$ and $b = y$:

$$u(x,y) = \mathrm{Re}(2z + \sin z) \;=\; 2\,\mathrm{Re}(z) + \mathrm{Re}(\sin z) = 2x + \sin x\cosh y\,,$$

$$v(x,y) = \mathrm{Im}(2z + \sin z) \;=\; 2\,\mathrm{Im}(z) + \mathrm{Im}(\sin z) = 2y + \cos x\sinh y\,.$$

(d) We use (17) with $z = x + iy$, so $a = x$ and $b = y$. Thus,

$$\begin{aligned} z\cos z &= (x + iy)(\cos x\cosh y - i\,\sin x\sinh y) \\ &= x\cos x\cosh y + y\sin x\sinh y + i(y\cos x\cosh y - x\sin x\sinh y)\,. \end{aligned}$$

Hence,

$$u(x,y) = \mathrm{Re}(z\cos z) \;=\; x\cos x\cosh y + y\sin x\sinh y\,,$$

$$v(x,y) = \mathrm{Im}(z\cos z) \;=\; y\cos x\cosh y - x\sin x\sinh y\,.$$

4. Consider first the complex cosine function. From (17),

$$\cos z = \cos x\cosh y - i\,\sin x\sinh y\,.$$

Thus,

$$\begin{aligned} |\cos z| &= \left(\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\right)^{1/2} \\ &= \left[\cos^2 x\cosh^2 y + (1 - \cos^2 x)\sinh^2 y\right]^{1/2} \\ &= \left[\cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y\right]^{1/2} \\ &= \left(\cos^2 x + \sinh^2 y\right)^{1/2}. \end{aligned}$$

Since

$$\sinh y = \frac{e^y - e^{-y}}{2}\,,$$

we see that

$$\lim_{y\to\infty} |\cos z| \to \tfrac{1}{2}e^y \to \infty\,.$$

An analogous argument shows that the complex sine function is also unbounded.

5. (a) With $2i = 2\,e^{\frac{1}{2}i\pi}$, we have

$$\ln(2i) = \ln\left(2\,e^{\frac{1}{2}i\pi}\right) = \ln 2 + \tfrac{1}{2}i\pi\,.$$

(b) With $-3-3\,i = 3\sqrt{2}\,e^{i\theta}$, where

$$\cos\theta = \sin\theta = -\frac{3}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}\,,$$

so $\theta = -\frac{3}{4}\pi$, we have

$$\ln(-3-3\,i) = \ln\left(3\sqrt{2}e^{i\theta}\right) = \ln 3\sqrt{2} - \frac{3\pi i}{4}\,.$$

(c)

$$\ln(4\,e^{\frac{1}{4}i\pi}) = \ln 4 + \frac{\pi\,i}{4}\,.$$