1st-Year Mathematics: Complex Analysis

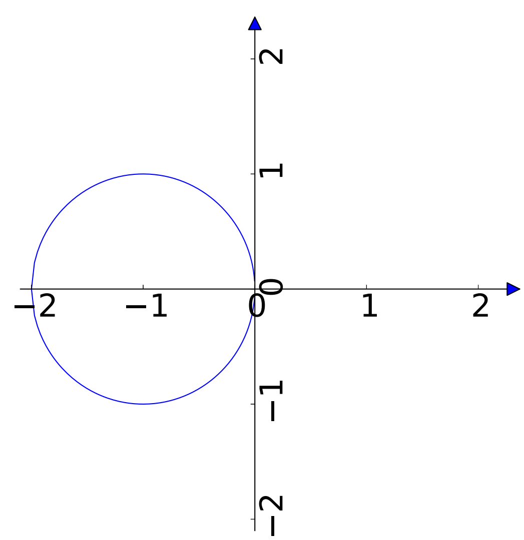
Solutions to Problem Sheet 1 2017

Tutorial

1. Given that $z = re^{i\theta}$, we have
$$(z^2)^* = \left[\left(re^{i\theta}\right)^2\right]^* = \left(r^2 e^{2i\theta}\right)^* = r^2 e^{-2i\theta} = \left(re^{-i\theta}\right)^2 = (z^*)^2 .$$

2. (a) Suppose z_0 is a solution of $az^2 + bz + c = 0$, and therefore also of $z^2 + b'z + c' = 0$ where $b' = b/a$ and $c' = c/a$ and $a \neq 0$. Given that z_0^* is also a solution implies that
$$(z - z_0)(z - z_0^*) = 0 ,$$
or
$$z^2 - (z_0 + z_0^*)z + |z_0|^2 = 0 ,$$
showing that b' and c' must be real. Conversely, if a, b, and c are real, then the conjugate of $az^2 + bz + c = 0$ is
$$a(z^2)^* + bz^* + c = 0 ,$$
or
$$a(z^*)^2 + bz^* + c = 0 ,$$
showing that z^* is also a solution.

(b) The roots of the polynomial are
$$z_\pm = \frac{1}{2t}\left(-1 \pm \sqrt{1 - 4t}\right)$$
which for $t > \frac{1}{4}$ has complex roots,
$$z_\pm = -\frac{1}{2t} \pm i\frac{1}{2t}\sqrt{4t - 1} .$$
The solutions will map out a circle centered at $z = -1$ in the complex plane. (To see this either show that $x = \text{Re}(z), y = \text{Im}(z)$ satisfies $(x+1)^2 + y^2 = 1$, or that $|z+1|^2 = 1$.)



(c) Rotating the figure through 90° is equivalent to multiplying both solutions by $e^{i\pi/2} = i$, thus

$$
\begin{aligned}
z_{\pm} &= i\left(-\frac{1}{2t} \pm i\frac{1}{2t}\sqrt{4t-1}\right) \\
&= \pm\frac{1}{2t}\sqrt{4t-1} - \frac{i}{2t}\,.
\end{aligned}
$$

Forming the polynomial by multiplying together the two solutions gives

$$
\begin{aligned}
p &= (z - z_+)\,(z - z_-) \\
&= \left(z - \frac{1}{2t}\sqrt{4t-1} + \frac{i}{2t}\right)\left(z + \frac{1}{2t}\sqrt{4t-1} + \frac{i}{2t}\right) \\
&= -\frac{1}{t}(-tz^2 - iz + 1)\,,
\end{aligned}
$$

thus reducing $p = 0$ to $-tz^2 - iz + 1 = 0$. So one solution is $a(t) = -t$, $b = -i$ and $c = 1$. All other solutions are multiples of this set. The solutions that were conjugate pairs for the original equation are not conjugate pairs for the new equation as the coefficients of the latter are no longer real. An arbitrary rotation of angle θ can be made by substituting $z_{\pm}$ with $e^{i\theta} z_{\pm}$.

Problems

1. For a complex number $a + ib$, in which a and b are real, the real and imaginary parts are given by $\mathrm{Re}(a+ib) = a$ and $\mathrm{Im}(a+ib) = b$, respectively. Thus,

   (a) $\mathrm{Re}(8+3i) = 8\,, \qquad \mathrm{Im}(8+3i) = 3\,.$

   (b) $\mathrm{Re}(4-15i) = 4\,, \qquad \mathrm{Im}(4-15i) = -15\,.$

   (c) $\mathrm{Re}(\cos\theta - i\sin\theta) = \cos\theta\,, \qquad \mathrm{Im}(\cos\theta - i\sin\theta) = -\sin\theta\,.$

   (d) $i^2 = -1. \quad \mathrm{Re}(i^2) = -1\,, \qquad \mathrm{Im}(i^2) = 0\,.$

   (e) $i(2-5i) = 5+2i. \qquad \mathrm{Re}(5+2i) = 5\,, \qquad \mathrm{Im}(5+2i) = 2\,.$

   (f) $(1+2i)(2-3i) = 2-3i+4i+6 = 8+i. \qquad \mathrm{Re}(8+i) = 8\,, \qquad \mathrm{Im}(8+i) = 1\,.$

2. Applying the rules for the multiplication and division of complex numbers yields:

(a) $(5-i)(2+3i) = 10+15i-2i+3 = 13+13i$.

(b) $(3-4i)(3+4i) = 9+12i-12i+16 = 25$.

(c) $(1+2i)^2 = (1+2i)(1+2i) = 1+2i+2i-4 = -3+4i$.

(d) $\dfrac{10}{4-2i} = \dfrac{10}{4-2i} \times \dfrac{4+2i}{4+2i} = \dfrac{40+20i}{16+8i-8i+4} = \dfrac{40+20i}{20} = 2+i$.

(e) $\dfrac{3-i}{4+3i} = \dfrac{3-i}{4+3i} \times \dfrac{4-3i}{4-3i} = \dfrac{12-9i-4i-3}{16-12i+12i+9} = \dfrac{9-13i}{25} = \dfrac{9}{25} - i\dfrac{13}{25}$.

(f) $\dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{-i}{-i} = -i$.

3. We have that $z = (5+7i)(5+bi) = 25+5bi+35i-7b$.

   (a) If b and z are both real, then the imaginary parts of both quantities vanish. Thus,

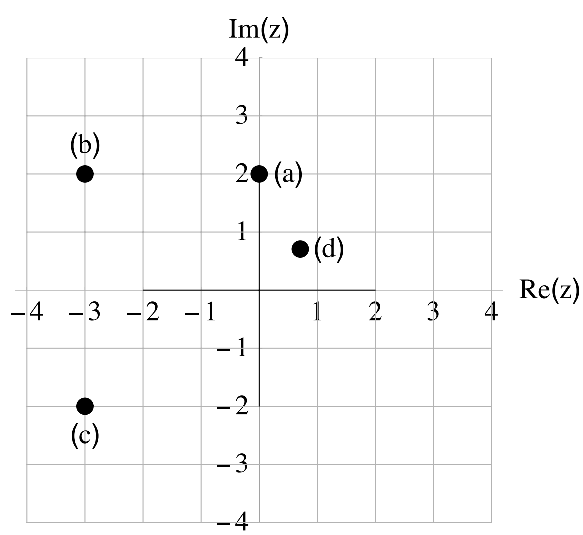
   $\mathrm{Im}(z) = 35+5b = 0$, so $b = -7$.

   (b) If $\mathrm{Im}(b) = \frac{4}{5}$, and z is pure imaginary, then the real part of z vanishes:

   $\mathrm{Re}(z) = 25+5\left[i\,\mathrm{Im}(b)\right]i - 7\,\mathrm{Re}(b) = 25-4-7\,\mathrm{Re}(b) = 21-7\,\mathrm{Re}(b) = 0,$

   so $\mathrm{Re}(b) = 3$.

4. The graphical representation of the complex number $z = x+iy$ is the point (x, y) on a set of axes where the x-axis corresponds to the real part of the complex number and the y-axis the imaginary part. The required points are



5. For a complex number $z = x+iy$, the polar form is $z = re^{i\theta}$, where

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

(a) $z = i$. We have that $x = 0$ and $y = 1$, so $r = 1$, and $\theta = \frac{1}{2}\pi$. Thus $i = e^{i\pi/2}$.

(b) $z = -i$. We have that $x = 0$ and $y = -1$, so $r = 1$, and $\theta = \frac{3}{2}\pi$. Thus $-i = e^{3i\pi/2}$.

(c) $z = 1 + i$. We have that $x = 1$ and $y = 1$, so $r = \sqrt{2}$, and $\theta = \frac{1}{4}\pi$.

Thus $1 + i = \sqrt{2}\, e^{i\pi/4}$.

(d) $z = 1 - i\sqrt{3}$. We have that $x = 1$ and $y = -\sqrt{3}$, so $r = 2$, and

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{1}{3}\pi$. Thus $1 - i\sqrt{3} = 2\, e^{-i\pi/3}$.

6. A complex number $z = re^{i\theta}$ can be written as $z = r\cos\theta + i r\sin\theta$. Thus,

(a) $e^{-3\pi i/4} = \cos\left(\frac{3}{4}\pi\right) - i\sin\left(\frac{3}{4}\pi\right) = -\dfrac{1+i}{\sqrt{2}} = -\dfrac{1}{2}\sqrt{2} - i\dfrac{1}{2}\sqrt{2}.$

(b) $e^{5\pi i/4} = \cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right) = -\dfrac{1+i}{\sqrt{2}} = -\dfrac{1}{2}\sqrt{2} - i\dfrac{1}{2}\sqrt{2}.$

(c) $3\,e^{i} = 3\cos 1 + i\sin 1\,.$

(d) $\dfrac{1}{\sqrt{3}\,e^{\pi i/3}} = \dfrac{\sqrt{3}}{3}\,e^{-\pi i/3} = \dfrac{\sqrt{3}}{3}\cos\left(\frac{1}{3}\pi\right) - \dfrac{i\sqrt{3}}{3}\sin\left(\frac{1}{3}\pi\right) = \dfrac{\sqrt{3}}{6} - \dfrac{i}{2}\,.$