Complex Analysis (Prof. M. McCall)
Multiple Choice Sheet 2

In this problem sheet we will look at how to translate a physical description of a problem into a differential equation.

1. A loan has an amount P outstanding. The interest rate is r percent per year and the loan is paid off at a constant rate of Q. Under the assumption that both interest is added and money paid off continuously, which of the equations below best describes how P changes with time. [3 marks]

   (a)
   $$\frac{dP}{dt} = \frac{r}{100}P - Q\,.$$
   (b)
   $$\frac{dP}{dt} = 100rP + Q\,.$$
   (c)
   $$\frac{dP}{dt} = \exp(Q/P) + \frac{r}{100}P\,.$$
   (d)
   $$\frac{dP}{dt} = \log(rP/Q) - Q\,.$$

2. Assume that a lake bed has the shape of a cone turned upside down. Thus for any given depth h in the middle of the lake, the surface is circular with a radius αh, where α is a constant. Water runs into the lake at a rate n_i, $[n_i] = \mathrm{m^3s^{-1}}$ and out at a rate $n_o h$, $[n_o] = \mathrm{m^2s^{-1}}$. Evaporation from the lake surface is at a rate of E per unit area, $[E] = \mathrm{ms^{-1}}$.

   (a) Which of the following is the differential equation satisfied by the depth in the centre of the lake as a function of time. [3 marks]

   i.
   $$\frac{dh}{dt} = \frac{n_i}{\pi\alpha^2h^2} - \frac{n_o}{\pi\alpha^2h} + \frac{E}{\pi\alpha}\,.$$
   ii.
   $$\frac{dh}{dt} = \frac{n_i}{\pi\alpha^2h^2} - \frac{n_o}{\pi\alpha^2h} - E\,.$$
   iii.
   $$\frac{dh}{dt} = \frac{n_i}{\pi^2\alpha^4h^2} - \frac{n_o}{\pi^2\alpha^4h} - E\,.$$

   (b) For large time a steady state is approached. For a given non-zero n_i, n_o and E, which of the following expressions describes the depth of the lake at very large times. [2 marks]

   i.
   $$h = \frac{n_o}{2E\pi\alpha^2}\left(\sqrt{1+\frac{4E\pi\alpha^2n_i}{n_o^2}} - 1\right)$$
   ii.
   $$h = \frac{\sqrt{E\pi\alpha^2n_i}}{En_0\pi\alpha^2}$$

3. Bismuth(210) can decay into Lead(206) in two different ways. Either by a β decay (yielding Polonium) followed by an α decay (yielding Thallium) or the other way around. Let $k_\beta^{\mathrm{Bi}(210)}$ denote the decay constant for the β decay of Bismuth(210) and use an equivalent notation for the other decays. Write down a system of coupled differential equations that expresses the amount of each isotope involved. [2 marks]

(a)

$$\frac{dN^{\mathrm{Bi}(210)}}{dt} = -(k_\beta^{\mathrm{Bi}(210)} + k_\alpha^{\mathrm{Bi}(210)})N^{\mathrm{Bi}(210)}$$
$$\frac{dN^{\mathrm{Po}(210)}}{dt} = k_\beta^{\mathrm{Po}(210)}N^{\mathrm{Bi}(210)} - k_\alpha^{\mathrm{Bi}(210)}N^{\mathrm{Po}(210)}$$
$$\frac{dN^{\mathrm{Tl}(206)}}{dt} = k_\alpha^{\mathrm{Tl}(210)}N^{\mathrm{Bi}(210)} - k_\beta^{\mathrm{Bi}(206)}N^{\mathrm{Tl}(206)}$$
$$\frac{dN^{\mathrm{Pb}(206)}}{dt} = k_\alpha^{\mathrm{Tl}(210)}N^{\mathrm{Po}(210)} + k_\beta^{\mathrm{Po}(206)}N^{\mathrm{Tl}(206)}$$

(b)

$$\frac{dN^{\mathrm{Bi}(210)}}{dt} = -k_\beta^{\mathrm{Bi}(210)}N^{\mathrm{Po}(210)} - k_\alpha^{\mathrm{Bi}(210)}N^{\mathrm{Tl}(206)}$$
$$\frac{dN^{\mathrm{Po}(210)}}{dt} = k_\beta^{\mathrm{Bi}(210)}N^{\mathrm{Po}(210)} - k_\alpha^{\mathrm{Po}(210)}N^{\mathrm{Pb}(206)}$$
$$\frac{dN^{\mathrm{Tl}(206)}}{dt} = k_\alpha^{\mathrm{Bi}(210)}N^{\mathrm{Tl}(206)} - k_\beta^{\mathrm{Tl}(206)}N^{\mathrm{Pb}(206)}$$
$$\frac{dN^{\mathrm{Pb}(206)}}{dt} = k_\alpha^{\mathrm{Po}(210)}N^{\mathrm{Po}(210)} + k_\beta^{\mathrm{Tl}(206)}N^{\mathrm{Tl}(206)}$$

(c)

$$\frac{dN^{\mathrm{Bi}(210)}}{dt} = -(k_\beta^{\mathrm{Bi}(210)} + k_\alpha^{\mathrm{Bi}(210)})N^{\mathrm{Bi}(210)}$$
$$\frac{dN^{\mathrm{Po}(210)}}{dt} = k_\beta^{\mathrm{Bi}(210)}N^{\mathrm{Bi}(210)} - k_\alpha^{\mathrm{Po}(210)}N^{\mathrm{Po}(210)}$$
$$\frac{dN^{\mathrm{Tl}(206)}}{dt} = k_\alpha^{\mathrm{Bi}(210)}N^{\mathrm{Bi}(210)} - k_\beta^{\mathrm{Tl}(206)}N^{\mathrm{Tl}(206)}$$
$$\frac{dN^{\mathrm{Pb}(206)}}{dt} = k_\alpha^{\mathrm{Po}(210)}N^{\mathrm{Po}(210)} + k_\beta^{\mathrm{Tl}(206)}N^{\mathrm{Tl}(206)}$$