Displacement-time graphs

1. **Displacement-time** graphs.

Any point on such a graph has coordinates \((t,s)\), in which \(s\) is the displacement after a time \(t\).

**Worked Example 1.**

Figure 1 shows the displacement-time graph for a tennis ball which is thrown vertically up in the air from a player’s hand and then falls to the ground.

The graph illustrates the motion of the ball.
Taking the origin to be the starting point, the ball moves upwards (in the positive direction) to a maximum height of 2.5 m above the origin. It stops momentarily and then drops to the ground, which is 1.5 m below the initial release.
Consequently, the velocity of the ball changes from positive (on the way up) to 0 at the top and then to negative (on the way down).

![Displacement-time graph](image)

*Figure 1*

**The gradient of a displacement-time graph is velocity**

Example problems:

1. A bus travels along a straight road for 600 m. It travels at a constant velocity for the whole journey, which takes 90 s. Sketch the displacement-time graph. What was the velocity of the bus?

2. A snooker ball moves in a straight line with a constant speed of \(u\) m s\(^{-1}\). It hits the cushion directly after a time \(t_1\) and rebounds along the same path with a constant speed of \((u - 0.2)\) m s\(^{-1}\). Sketch the displacement-time graph. (Assume \(u > 0.2\))
1. Velocity $= \frac{600}{90} = 6.7 \text{ m s}^{-1}$

2. It is important to realise that the gradient between $t = 0$ and $t = t_1$ is steeper than that after $t_1$, to reflect the greater velocity before impact with the cushion.
**Example 1**

A cyclist rides in a straight line for 20 minutes. She waits for half an hour, then returns in a straight line to her starting point in 15 minutes. This is a displacement–time graph for her journey.

a. Work out the average velocity for each stage of the journey in km h$^{-1}$.

b. Write down the average velocity for the whole journey.

c. Work out the average speed for the whole journey.

---

**Solution:**

a. Journey from O to A: time = 20 mins; displacement = 5 km

   Average velocity = \( \frac{5}{20} = 0.25 \text{ km min}^{-1} \)

   \( 0.25 \times 60 = 15 \text{ km h}^{-1} \)

   Journey from A to B: no change in displacement
   so average velocity = 0

   Journey from B to C: time = 15 mins; displacement = −5 km

   Average velocity = \( \frac{-5}{15} = -\frac{1}{3} \text{ km min}^{-1} \)

   \( -\frac{1}{3} \times 60 = -20 \text{ km h}^{-1} \)

b. The displacement for the whole journey is 0 so average velocity is 0.

c. Total time = 65 mins

   Total distance travelled is 5 + 5 = 10 km

   Average speed = \( \frac{10}{65} = \frac{2}{13} \text{ km min}^{-1} \)

   \( \frac{2}{13} \times 60 = 9.2 \text{ km h}^{-1} \) (2 s.f.)

---

To convert from km min$^{-1}$ to km h$^{-1}$ multiply by 60.

A horizontal line on the graph indicates the cyclist is stationary.

The cyclist starts with a displacement of 5 km and finishes with a displacement of 0 km, so the change in displacement is −5 km, and velocity will be negative.

At C the cyclist has returned to the starting point.

The cyclist has travelled 5 km away from the starting point and then 5 km back to the starting point.
Ex 9A p132 AS Textbook

Qt

1 This is a displacement–time graph for a car travelling along a straight road. The journey is divided into 5 stages labelled A to E.
   a Work out the average velocity for each stage of the journey.
   b State the average velocity for the whole journey.
   c Work out the average speed for the whole journey.

---

1 a $A$ displacement = 40 km, time = 0.5 h and $\frac{40}{0.5} = 80$
   So the average velocity is 80 km h$^{-1}$.

   $B$ displacement = 20 km, time = 0.5 h and $\frac{20}{0.5} = 40$
   So the average velocity is 40 km h$^{-1}$.

   $C$ displacement = 0 km, time = 0.5 h and $\frac{0}{0.5} = 0$
   So the average velocity is 0 km h$^{-1}$.

   $D$ displacement = 40 km, time = 1 h and $\frac{40}{1} = 40$
   So the average velocity is 40 km h$^{-1}$.

   $E$ displacement = −100 km, time = 1.5 h and $\frac{100}{1.5} = -66.7$ (to 3 s.f.)
   So the average velocity is -66.7 km h$^{-1}$.

b The average velocity for the whole journey is 0 km h$^{-1}$ as the overall displacement is 0 km.

c Total distance travelled = 200 km
   Total time taken = 4 h
   average speed = $\frac{200}{4} = 50$ km h$^{-1}$
2 Khalid drives from his home to a hotel. He drives for 2\(\frac{1}{2}\) hours at an average velocity of 60 km h\(^{-1}\). He then stops for lunch before continuing to his hotel. The diagram shows a displacement–time graph for Khalid’s journey.

a Work out the displacement of the hotel from Khalid’s home.

b Work out Khalid’s average velocity for his whole journey.

\[
\begin{align*}
2 \text{ a } & \text{ For first section of the journey: average velocity } = 60 \text{ km h}^{-1}, \text{ time taken } = 2.5 \text{ h} \\
& \text{displacement } = 2.5 \times 60 = 150 \text{ km} \\
& \text{This is 6 squares on the vertical axis, so one square is } \frac{150}{6} = 25 \text{ km} \\
& \text{total displacement shows as 7.5 squares } = 7.5 \times 25 = 187.5 \text{ km} \\
\text{b } & \text{Time for whole journey } = 3.75 \text{ h} \\
& \text{average velocity } = \frac{187.5}{3.75} = 50 \text{ km h}^{-1}
\end{align*}
\]
Q3

Sarah left home at 10:00 and cycled north in a straight line. The diagram shows a displacement–time graph for her journey.

a Work out Sarah’s velocity between 10:00 and 11:00.
On her return journey, Sarah continued past her home before returning.
b Estimate the time that Sarah passed her home.
c Work out Sarah’s velocity for each of the last two stages of her journey.
d Calculate Sarah’s average speed for her entire journey.

3 a displacement = 12 km, time = 1 h
average velocity = \frac{12}{1} = 12 \text{ km h}^{-1}

b Sarah passed her home at 12:45.

c For the penultimate stage: displacement = -12 + (3) = -15 \text{ km}, time = 1.5 h
average velocity = \frac{-15}{1.5} = -10 \text{ km h}^{-1}

For the final stage: displacement = 3 \text{ km}, time = 1 h
average velocity = \frac{3}{1} = 3 \text{ km h}^{-1}

3 d Total distance travelled = 30 km
Total time taken = 4 h
average speed = \frac{30}{4} = 7.5 \text{ km h}^{-1}
4 A ball is thrown vertically up in the air and falls to the ground. This is a displacement–time graph for the motion of the ball.

a Find the maximum height of the ball and the time at which it reaches that height.

b Write down the velocity of the ball when it reaches its highest point.

c Describe the motion of the ball:

i from the time it is thrown to the time it reaches its highest point

ii after reaching its highest point.

---

4 a Reading from the graph:

maximum height = 2.5 m

time taken to reach this = 0.75 s

b When it reaches the highest point, the velocity of the ball is 0 m s\(^{-1}\).

c i The velocity of the ball is positive (upwards) and decreases (the ball is decelerating) until it reaches 0 at the highest point.

ii The velocity of the ball is negative (downwards), and increases (the ball is accelerating) until it hits the ground at the same speed at which it was launched.
Q5

a) Sketch a displacement-time graph for a ball thrown vertically upwards with an initial velocity $\sqrt{5g} \text{ m/s}$.

b) Sketch a velocity-time graph to show this motion.

The ball is modelled as a particle moving freely under the influence of gravity alone.

c) Explain two additional factors which may be taken into account in a refinement of the above model to make it more realistic.

Solution overleaf.
Q5 Solution

a) The s-t graph above shows the displacement increasing with time to its maximum value (the maximum height of the ball), then decreasing back to 0 (when the ball hits the ground).
Remember that the gradient is the velocity. And in this example the velocity is not constant, therefore the gradient of the graph will not be constant; instead, the gradient will be curved i.e. the line will be curved.

b) The v-t graph demonstrates that the velocity of the ball decreases to 0. Then the velocity increases back up to the initial velocity, however the ball is now travelling in the opposite direction so velocity will take a negative value, hence the graph.

Note that the gradient of this graph represents acceleration ($\frac{v_f - v_i}{t}$). The ball has an acceleration of $-g$ ms$^{-2}$ (because gravity acts downwards all of the time and the question states that upwards is the positive direction). Therefore, the gradient of this graph will be a straight line from $v = +7$ to $v = -7$.

c) Air resistance and rotation of ball. Air resistance will oppose the direction of travel, and rotational forces will come into play if our ball is not modelled in this way. Thus, using $\Sigma F = ma$, we will find a different (varying) value for acceleration of the ball. (Assuming the ball is a particle we say that $\Sigma F = -mg$, so $-mg = ma$, so $-g = a$).
Mixed Exercise 9 AS Textbook

Q1

1. A car accelerates in a straight line at a constant rate, starting from rest at a point A and reaching a velocity of 45 km h⁻¹ in 20 s. This velocity is then maintained and the car passes a point B 3 minutes after leaving A.
   a. Sketch a velocity–time graph to illustrate the motion of the car.
   b. Find the displacement of the car from its starting point after 3 minutes.

Solution overleaf.
1a \quad 45 \text{ km h}^{-1} = \frac{45 \times 1000}{3600} \text{ m s}^{-1} = 12.5 \text{ m s}^{-1}

3 \text{ min} = 180 \text{ s}

\[v(\text{m s}^{-1})\]

\[\begin{array}{c}
    0 \\
    20 \\
    180 \\
\end{array}
\]

\[t(\text{s})\]

b \quad s = \frac{1}{2} (a + b)h

= \frac{1}{2} (160 + 180) \times 12.5 = 2125

The distance from A to B is 2125 m.
Q2

A particle is moving on an axis Ox. From time $t = 0$ s to time $t = 32$ s, the particle is travelling with constant velocity $15$ m s$^{-1}$. The particle then decelerates from $15$ m s$^{-1}$ to rest in $T$ seconds.

a Sketch a velocity–time graph to illustrate the motion of the particle.

The total distance travelled by the particle is 570 m.

b Find the value of $T$.

c Sketch a displacement–time graph illustrating the motion of the particle.
2 a

\[ v(t) \text{ (m/s)} \]

\[
\begin{array}{c}
15 \\
O \\
32 \\
32 + T \\
\end{array}
\]

b \[ s = \frac{1}{2}(a + b)h \]

\[
570 = \frac{1}{2}(32 + 32 + T) \times 15
\]

\[
\frac{15}{2} (T + 64) = 570
\]

\[
T + 64 = \frac{570 \times 2}{15} = 76
\]

\[
T = 76 - 64 = 12
\]

c At \ t = 32, \ s = 32 \times 15 = 480

At \ t = 44, \ s = 480 + \text{area of the triangle}

\[
= 480 + \frac{1}{2} \times 12 \times 15 = 570
\]
Exercise 11C AS Textbook Q5

5 A particle \( P \) starts at the origin \( O \) at time \( t = 0 \) and moves along the \( x \)-axis. At time \( t \) seconds the distance of the particle, \( s \) m, from the origin is given by:

\[ s = \frac{9t^2}{2} - t^3, \quad 0 \leq t \leq 4.5 \]

a Sketch a displacement–time graph for the motion of \( P \).
b Hence justify the restriction \( 0 \leq t \leq 4.5 \).
c Find the maximum distance of the particle from \( O \).
d Find the magnitude of the acceleration of the particle at this point.

Solution overleaf.
5 a \[ s = \frac{9t^2}{2} - t^3 \]
\[ = t^2(4.5 - t) \]

Displacement is zero when \( t = 0 \) and \( t = 4.5 \).
The graph touches the time axis at \( t = 0 \) and crosses it at \( t = 4.5 \).

Graph only shown for \( 0 \leq t \leq 4.5 \), as this is range over which equation is valid.
The curve is cubic, so not symmetrical.

\[ \begin{array}{c}
\text{Displacement, } s \\
\text{Time, } t
\end{array} \]

b For values of \( t > 4.5 \), \( s \) is negative. However \( s \) is a distance and can only be positive.

c \[ s = \frac{9t^2}{2} - t^3 \]
\[ \frac{ds}{dt} = 9t - 3t^2 \]
\[ \frac{ds}{dt} = 0 \text{ when} \]
\[ 9t - 3t^2 = 0 \]
\[ 3t(3 - t) = 0 \]

The turning points are at \( t = 0 \) and \( t = 3 \).
\( s = 0 \) when \( t = 0 \), so maximum distance occurs when \( t = 3 \).

When \( t = 3 \), using factorised form of the equation of motion:
\[ s = 3^2(4.5 - 3) = 9 \times 1.5 = 13.5 \]

The maximum distance of \( P \) from \( O \) is 13.5 m.

5 d \[ v = \frac{ds}{dt} = 9t - 3t^2 \]
\[ a = \frac{dv}{dt} = 9 - 6t \]

When \( t = 3 \),
\[ a = 9 - 6 \times 3 = -9 \]
The magnitude of the acceleration of \( P \) at the maximum distance is 9 m s\(^{-2}\).
3 An electric train starts from rest at a station $A$ and moves along a straight level track. The train accelerates uniformly at $0.4 \text{ m s}^{-2}$ to a speed of $16 \text{ m s}^{-1}$. The speed is then maintained for a distance of $2000 \text{ m}$. Finally the train retards uniformly for $20 \text{ s}$ before coming to rest at a station $B$. For this journey from $A$ to $B$,

a find the total time taken

b find the distance from $A$ to $B$

c sketch the displacement–time graph, showing clearly the shape of the graph for each stage of the journey.
3 a Let the time for which the train accelerates be $t_1$ s and the time for which it travels at a constant speed be $t_2$ s.

During acceleration

\[ v = u + at \]

\[ 16 = 0 + 0.4t_1 \Rightarrow t_1 = \frac{16}{0.4} = 40 \]

At constant speed

\[ 2000 = 16 \times t_2 \Rightarrow t_2 = \frac{2000}{16} = 125 \]

The total time is $(t_1 + t_2 + 20) s = (40 + 125 + 20) s = 185 s$

b \quad s = \frac{1}{2} (a + b) h$

\[ = \frac{1}{2} (125 + 185) \times 16 = 2480 \]

$AB = 2480 \text{ m}$

c

[Diagram showing x(t) graph with points marked at x=2480, x=2320, x=320, t=40, t=165, t=185]
References:
www.mathcentre.ac.uk p1-2
Pearsonactivelearn AS Statistics and Mechanics textbook p2-17