Orthogonal Curvilinear Coordinates

1 Introduction

Many problems in physics have a central point or axis. For these, cartesian \((x, y, z)\) coordinates can be tedious, and it is natural to introduce a coordinate system that reflects the shapes and symmetries of the problem. Examples include cylindrical and spherical polar coordinates, which we shall explore further here, parabolic or hyperbolic coordinates, and others. Many are constructed so that the corresponding unit vectors \((\hat{i}, \hat{j}, \hat{k})\), \((\hat{\rho}, \hat{\phi}, \hat{k})\), etc., are orthogonal (i.e., perpendicular to one another). Since in these systems lines of constant components (e.g., constant \(r\)) are curved, we refer to such coordinate systems as “orthogonal curvilinear coordinates.” Below is a summary of the main aspects of two of the most important systems, cylindrical and spherical polar coordinates. Many of the steps presented take subtle advantage of the orthogonal nature of these systems.

You can find complementary material in both Riley et al., *Mathematical Methods for Physics and Engineering*, Sections 10.9 and 10.10, and in Boas, *Mathematical Methods in the Physics Sciences*, Chapter 10, Sections 6–9. These approaches tend to be more mathematical and general than the one given here.

You will not necessarily be expected to reproduce the calculations given below. However, much of it is quite instructive in terms of the way cylindrical and spherical polar coordinates work. Additionally, seeing how the forms of grad, div, and curl, together with line, surface and volume elements, are derived in different systems helps provide some insight into their interpretation.

2 Cylindrical Polar Coordinates

![Figure 1: Cylindrical polar coordinates and their volume element.](image)

To begin, let us recall some basics about cylindrical polar coordinates (see Figure 1). From this
2.1 Position Vector

The point $P$ indicated by the black dot in Figure 1 has position vector

$$
\mathbf{r} = \rho \hat{\rho} + z \hat{k}
$$

(4)

At first sight this might seem strange; where is the $\hat{\phi}$ component? But if you look at Figure 1 you can see that you do need to know three things to locate $P$: $\rho$, $z$ and $\hat{\rho} = \hat{\rho}(\phi)$. So the information about $\phi$ is contained in $\hat{\rho}$; different $\phi$ values give you $\hat{\rho}$ unit vectors that point in different directions.

A general vector field $\mathbf{B}$ will have three components $\mathbf{B}(\rho, \phi, z) = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{k}$; the position vector is special in this regard.

From Figure 1 or from (1)–(3) we see that

$$
\mathbf{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}
$$

(5)

$$
= \rho \left( \cos \phi \hat{i} + \sin \phi \hat{j} \right) + z \hat{k}
$$

(6)

$$
= \hat{\rho} \cos \phi \hat{i} + \sin \phi \hat{j} + z \hat{k}
$$

(7)

From which we deduce that

$$
\hat{\rho} = \cos \phi \hat{i} + \sin \phi \hat{j}
$$

(8)

You can see this either by constructing the necessary trigonometry or simply note that $\hat{\rho}$ must lie in the $x - y$ plane and be perpendicular to $\hat{\rho}$. Indeed, $(\rho, \phi)$ are just the plane polar coordinates $(r, \theta)$ in disguise.

Finally, from (8) we see that

$$
\frac{d\hat{\rho}}{d\phi} = \hat{\phi}
$$

(10)

2.2 Line Element

Now that we have established the representation of a position $\mathbf{r}$ we can proceed to consider a displacement $d\mathbf{r}$ from that position. Graphically with reference to Figure 1, if we increment $\rho$ by an amount $d\rho$ then the vector displacement would be $d\rho \hat{\rho}$. Incrementing $\phi$ by an amount $d\phi$ would be a vector displacement $\rho d\phi \hat{\phi}$. And incrementing $z$ by $dz$ would be a vector displacement $dz \hat{k}$. An arbitrary displacement would be the vector sum of these, i.e.,

$$
d\mathbf{r} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{k}
$$

(11)

Interestingly, and perhaps reassuringly, you can get to (11) by taking the differential of $\mathbf{r}$ directly from (7):

$$
d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz
$$

(12)

$$
= d\rho \hat{\rho} + \rho \frac{\partial \phi}{\partial \phi} \hat{\phi} + dz \hat{k}
$$

(13)

$$
= d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{k}
$$

(14)

where we have made use of (10).

2.3 Surface Elements

To find a surface element $dS$ we would need to use the formal machinery we derived in lecture, namely if we have a surface $S$ parametrised by two variables $(u, v)$ so that a point on the surface is $\mathbf{r} = \mathbf{r}(u, v)$, then

$$
dS = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv
$$

(15)

Since $\hat{\rho}, \hat{\phi}, \hat{k}$ are mutually orthogonal, you could perform this cross product in cylindrical polars if that was convenient, i.e.,

$$
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \left| \begin{array}{ccc} \hat{\rho} & \hat{\phi} & \hat{k} \\ \frac{\partial \mathbf{r}}{\partial u} & \frac{\partial \mathbf{r}}{\partial v} & \frac{\partial \mathbf{r}}{\partial w} \\ \frac{\partial \mathbf{r}}{\partial u} & \frac{\partial \mathbf{r}}{\partial v} & \frac{\partial \mathbf{r}}{\partial w} \end{array} \right|
$$

(16)

Let’s look at the surfaces of the elemental volume shown in Figure 1. If we start with the bottom surface, defined by $z = \text{constant} = z_0$ say, then we can parametrise this surface by $\rho$ and $\phi$, and a point on this surface is described simply by (7) with constant $z$, i.e., $\mathbf{r}(\rho, \phi) = \rho \hat{\rho}(\phi) + z_0 \hat{k}$. So we have

$$
\frac{\partial \mathbf{r}}{\partial \rho} = \hat{\rho} + 0 \hat{\phi} + 0 \hat{k}; \quad \frac{\partial \mathbf{r}}{\partial \phi} = 0 \hat{\rho} + \rho \hat{\phi} + 0 \hat{k}
$$
again making use of (10). Thus

\[ dS = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{k} \\ \rho & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} d\rho \ d\phi = \rho \ d\rho \ d\phi \ \hat{k} \]  

(17)

Actually, you should be able to look at Figure 1, see that this face is roughly rectangular and has an area \( \rho \ d\phi \times d\rho \) and that its normal is \( \hat{k} \) and write down immediately (17).

In a similar way it is now easy to see (I hope) that the surface element for the flat-sided side face nearest the viewer is

\[ dS = d\rho \ dz \ \hat{\phi} \]  

(18)

while that of the inner curved face is

\[ dS = \rho \ d\phi \ dz \ \hat{\rho} \]  

(19)

Note that for all of these pieces of surface, there is an ambiguity of a \( \pm \) sign depending on the application at hand. For example, if we were interested in the surface elements with normals pointing out of the volume element, we would need to take the negative of all the above \( dS \) expressions for some faces (see Figure 2 below).

### 2.4 Volume Elements

Again taking advantage of the orthogonality of the unit vectors and looking at Figure 1 we can immediately write down the volume of the differential volume depicted there. This volume is simply the product of the three sides of the (pseudo-)rectangular volume, i.e.,

\[ dV = \rho \ d\phi \ d\rho \ dz \]  

(20)

You can also deduce this by calculating the Jacobian \( \frac{\partial (x, y, z)}{\partial (\rho, \phi, z)} \) as we did in lecture.

### 2.5 Gradient Operator

Let’s now turn our attention to the vector calculus operators, beginning with the gradient. We will do this by constructing an exact differential of a scalar function \( f \) by writing

\[ df = \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \phi} d\phi + \frac{\partial f}{\partial z} dz \]  

(21)

\[ \nabla f = \cdot \, dr \]  

(22)

Figure 2: Volume element in cylindrical polars with surface elements marked for application of the Divergence Theorem

If we now express \( \nabla f \) in components, i.e., \( \nabla f = (\nabla f)_\rho \hat{\rho} + (\nabla f)_\phi \hat{\phi} + (\nabla f)_z \hat{k} \), and make use of (7) for \( df \) this gives

\[ df = \left( (\nabla f)_\rho \hat{\rho} + (\nabla f)_\phi \hat{\phi} + (\nabla f)_z \hat{k} \right) \cdot 
\]  

\[ 
\quad \left( d\rho \hat{\rho} + \rho \ d\phi \hat{\phi} + dz \hat{k} \right) 
\]

\[ = (\nabla f)_\rho \ d\rho + (\nabla f)_\phi \ d\phi + (\nabla f)_z \ dz \]  

(23)

Now \( d\rho, \ d\phi, \) and \( dz \) are all independent, so their coefficients in (21) and (23) must be equal. Thus \( (\nabla f)_\rho = \frac{\partial f}{\partial \rho}, \ (\nabla f)_\phi = \frac{\partial f}{\partial \phi}, \) and \( (\nabla f)_z = \frac{\partial f}{\partial z} \) so we reach

\[ \nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{k} \]  

(24)

### 2.6 Divergence Operator

We can derive an expression for the divergence operator of a vector field \( \mathbf{B}(\rho, \phi, z) \) by applying the Divergence Theorem to the elemental volume shown in Figure 1 and expanded in Figure 2. The steps below are essentially the reverse of those used to derive the Divergence Theorem in the first place in cartesian coordinates. The underlying concepts are identical, although the algebra here is a bit more involved, essentially because the lengths of some of the sides of the volume element depend on the value of \( \rho \).
The Divergence Theorem states that
\[ \iiint_V \nabla \cdot \mathbf{B} \, dV = \iint_S \mathbf{B} \cdot d\mathbf{S} \]  
(25)
When applied to an elemental volume, we can remove the integration signs to reveal that
\[ \nabla \cdot \mathbf{B} = \sum_{i=1}^{6} \mathbf{B} \cdot d\mathbf{S}_i / dV \]  
(26)
If we consider \( d\mathbf{S}_1 \) and \( d\mathbf{S}_2 \) to begin with, we can see from Figure 2 that
\[
\begin{align*}
    d\mathbf{S}_1 &= -\rho \Delta \phi \Delta z \hat{\rho} \\
    d\mathbf{S}_2 &= +\rho (\rho + \Delta \rho) \Delta \phi \Delta z \hat{\rho}
\end{align*}
\]
so that
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_1 &= -B_{\rho}(\rho, \phi^*, z^*) \rho \Delta \phi \Delta z \\
    \mathbf{B} \cdot d\mathbf{S}_2 &= +B_{\rho}(\rho + \Delta \rho, \phi^*, z^*) (\rho + \Delta \rho) \Delta \phi \Delta z
\end{align*}
\]
where \( \phi^* \) and \( z^* \) denote the values of \( \phi \) and \( z \) for which \( B_{\rho} \) takes on its average value over \( d\mathbf{S}_1 \), and similarly for \( d\mathbf{S}_2 \). By virtue of the mean value theorem, \( \phi \leq \phi^* \leq (\phi + \Delta \phi) \), etc. Combining these two expressions then yields
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_1 + \mathbf{B} \cdot d\mathbf{S}_2 &= \\
    &= \left[ (\rho + \Delta \rho) B_{\rho}(\rho + \Delta \rho, , ) - \rho B_{\rho}(\rho, , ) \right] \Delta \rho \Delta \phi \Delta z
\end{align*}
\]
where we have suppressed \( \phi^* \), etc., quantities for the sake of brevity. Now as we let the volume shrink to differential proportions,
\[
\begin{align*}
    \Delta \rho &\rightarrow d\rho \\
    \Delta \phi &\rightarrow dz \\
    \phi^* &\rightarrow \phi \\
    \phi^{**} &\rightarrow \phi \\
    z^* &\rightarrow z \\
    z^{**} &\rightarrow z
\end{align*}
\]
It remains only to let \( \Delta \rho \) shrink to its limiting differential form:
\[
\begin{align*}
    \Delta \rho &\rightarrow d\rho \\
    (\rho + \Delta \rho) B_{\rho}(\rho + \Delta \rho, , ) - \rho B_{\rho}(\rho, , ) &\rightarrow \frac{\partial (\rho B_{\rho})}{\partial \rho} d\rho
\end{align*}
\]
so that (29) becomes
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_1 + \mathbf{B} \cdot d\mathbf{S}_2 &= \\
    &= \int_{\rho \Delta \phi}^{(\rho + \Delta \rho) B_{\rho}(\rho + \Delta \rho, , ) - \rho B_{\rho}(\rho, , )} \frac{\partial (\rho B_{\rho})}{\partial \rho} d\rho dz
\end{align*}
\]
where we have multiplied and divided by \( \rho \) in the first line to form the volume element \( \rho \, d\rho \, d\phi \, dz = dV \) as in (20).

In a similar way, \( d\mathbf{S}_3 \) and \( d\mathbf{S}_4 \) are in the \( \pm \phi \) direction and in this case both have area \( \Delta \rho \Delta \phi \). \( d\mathbf{S}_3 \) is at \( \phi \) while \( d\mathbf{S}_4 \) is at \( \phi + \Delta \phi \). So the analog to (29) is
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_3 + \mathbf{B} \cdot d\mathbf{S}_4 &= \\
    &= \left[ B_{\phi}(\phi + \Delta \phi, , ) - B_{\phi}(\phi, , ) \right] \Delta \rho \Delta \phi \Delta z
\end{align*}
\]
which leads to
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_3 + \mathbf{B} \cdot d\mathbf{S}_4 &= \\
    &= \frac{1}{\rho} \frac{\partial B_{\phi}}{\partial \phi} d\rho \, d\phi \, dz
\end{align*}
\]
Finally, \( d\mathbf{S}_{5,6} = \pm \Delta \rho \rho \Delta \phi \hat{k} \) so that
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_5 + \mathbf{B} \cdot d\mathbf{S}_6 &= \\
    &= \left[ B_{z}(\rho + \Delta \rho, z + \Delta z) - B_{z}(\rho, z) \right] \Delta \rho \Delta \phi
\end{align*}
\]
which leads to
\[
\begin{align*}
    \mathbf{B} \cdot d\mathbf{S}_5 + \mathbf{B} \cdot d\mathbf{S}_6 &= \\
    &= \frac{\partial B_{z}}{\partial z} d\rho \, d\phi \, dz
\end{align*}
\]
Putting (31), (34), and (37) into the summation in (26) yields the desired result, namely an expression for \( \nabla \cdot \mathbf{B} \) in cylindrical polar coordinates:
\[
\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial (\rho B_{\rho})}{\partial \rho} + \frac{1}{\rho} \frac{\partial B_{\phi}}{\partial \phi} + \frac{\partial B_{z}}{\partial z}
\]
\( \) (38)

2.7 Curl Operator

A similar approach to that in Section 2.6 can be applied to derive an expression for the curl of a vector field in cylindrical polar coordinates, this time starting from Stoke's Theorem:
\[
\oint_C \mathbf{B} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{B} \cdot d\mathbf{S}
\]
(39)
which for a differential surface element reduces to
\[
\sum_{i=1}^{n_{edges}} \mathbf{B} \cdot d\mathbf{r}_i = \nabla \times \mathbf{B} \cdot d\mathbf{S}
\]
(40)
We will pick our surface elements from the faces of the volume element at \( \rho, \phi, \) and \( z \) shown in Figure 2.
2.7.1 $\hat{\rho}$ Component

The face with normal $\hat{\rho}$ is shown in Figure 3. The four edges have $dr_1 = +\rho \Delta \phi \hat{\rho}$, $dr_2 = +\Delta z \hat{k}$, $dr_3 = -\rho \Delta \phi \hat{\rho}$, and $dr_4 = -\Delta z \hat{k}$ while $dS_\rho = \rho \Delta \phi \Delta z \hat{\rho}$. Thus (40) becomes

$$B \cdot dr_1 + B \cdot dr_2 + B \cdot dr_3 + B \cdot dr_4 =$$

$$B_\phi (\rho, \phi^*, z) \rho \Delta \phi + B_z (\rho, \phi + \Delta \phi, z^*) \Delta z -$$

$$-B_\phi (\rho, \phi^*, z + \Delta z) \rho \Delta \phi - B_z (\rho, \phi, z^*) \Delta z -$$

$$-[B_z (\rho, \phi + \Delta \phi, z^*) - B_z (\rho, \phi, z^*)] \Delta z -$$

$$-[B_\phi (\rho, \phi^*, z + \Delta z) - B_\phi (\rho, \phi^*, z)] \rho \Delta \phi$$

$$= (\nabla \times B)_\rho \rho \Delta \phi \Delta z$$  

with $\phi^*$, $z^*$ again denoting the value within the interval for which the relevant component of $B$ takes on its mean value. As we let this surface shrink to differential proportions, this reduces to:

$$(\nabla \times B)_\rho \rho d\phi dz = \left( \frac{\partial B_z}{\partial \phi} d\phi \right) dz - \left( \frac{\partial B_\phi}{\partial z} dz \right) \rho d\phi$$

$$= \left( \frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right) \rho d\phi dz$$  

(42)

from which we deduce that the $\rho$ component of $\nabla \times B$ is

$$(\nabla \times B)_\rho = \frac{1}{\rho} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}$$  

(43)

Figure 3: Line integration to determine the $\hat{\rho}$ component of $\nabla \times B$.

2.7.2 $\hat{\phi}$ Component

The other components of $\nabla \times B$ follow similarly. Figure 4 shows the $\hat{\phi}$ face. Be careful to ensure that you go around the edges in a right-handed sense with respect to $dS$, which we have chosen here to be in the $+\hat{\phi}$ direction, so that $dS = \Delta \rho \Delta z \hat{\phi}$. Evaluating (40) for this face yields

$$B \cdot dr_1 + B \cdot dr_2 + B \cdot dr_3 + B \cdot dr_4 =$$

$$B_z (\rho, \phi) \Delta z + B_\phi (\rho, z + \Delta z) \Delta \rho -$$

$$-B_z (\rho + \Delta \rho, \phi) \Delta z - B_\phi (\rho, z) \Delta \rho -$$

$$-[B_\phi (\rho, z + \Delta z) - B_\phi (\rho, z)] \Delta \rho -$$

$$-[B_z (\rho + \Delta \rho, z) - B_z (\rho, z)] \Delta z$$

$$= (\nabla \times B)_\phi \Delta \rho \Delta z$$  

(44)

Here for brevity and clarity I have omitted the dependencies which are either constant along a particular edge or evaluated at some "ed value along them while retaining the dependency that characterises which edge we are following. Letting the surface shrink to its differential form yields

$$(\nabla \times B)_\phi d\rho dz =$$

$$= \left( \frac{\partial B_\phi}{\partial \rho} d\rho \right) dz - \left( \frac{\partial B_z}{\partial \rho} d\rho \right) dz$$  

(45)

from which we can see that

$$(\nabla \times B)_\phi = \frac{\partial B_\phi}{\partial \rho} - \frac{\partial B_z}{\partial \rho}$$  

(46)

Figure 4: Line integration to determine the $\hat{\phi}$ component of $\nabla \times B$. 
2.7.3 \( \hat{k} \) Component

There remains only the \( \hat{k} \)-component of \( \nabla \times \mathbf{B} \) to calculate, by reference to Figure 5. Following the now-familiar pattern, \( dS_z = \Delta \rho \Delta \phi \hat{k} \) and

\[
\mathbf{B} \cdot d\mathbf{r}_1 + \mathbf{B} \cdot d\mathbf{r}_2 + \mathbf{B} \cdot d\mathbf{r}_3 + \mathbf{B} \cdot d\mathbf{r}_4 = \\
= B_\rho(\rho, \phi) \Delta \rho + B_\phi(\rho + \Delta \rho, \phi)(\rho + \Delta \rho) \Delta \phi - \\
- B_\rho(\rho, \phi + \Delta \phi) \Delta \phi - B_\phi(\rho, \phi) \rho \Delta \phi \\
= \left[ (\rho + \Delta \rho)B_\phi(\rho + \Delta \rho, \phi) - \rho B_\phi(\rho, \phi) \right] \Delta \phi \\
- \left[ B_\rho(\rho, \phi + \Delta \phi) - B_\rho(\rho, \phi) \right] \Delta \rho \\
= (\nabla \times \mathbf{B})_z \rho \Delta \rho \Delta \phi
\]

(47)

Again shrinking to differential size yields

\[
(\nabla \times \mathbf{B})_z = \frac{1}{\rho} \frac{\partial (\rho B_\phi)}{\partial \rho} d\phi - \frac{1}{\rho} \frac{\partial B_\rho}{\partial \phi} d\rho
\]

(48)

from which we deduce

\[
(\nabla \times \mathbf{B})_z = \frac{1}{\rho} \frac{\partial (\rho B_\phi)}{\partial \rho} d\phi - \frac{1}{\rho} \frac{\partial B_\rho}{\partial \phi}
\]

(49)

2.7.4 \( \nabla \times \mathbf{B} \)

If we now collect together (43), (46), and (49) we see that in cylindrical polar coordinates the curl takes the form

\[
\nabla \times \mathbf{B} = \left( \frac{1}{\rho} \frac{\partial B_\phi}{\partial \rho} - \frac{\partial B_\rho}{\partial z} \right) \hat{\rho} + \\
+ \left( \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} \right) \hat{\phi} + \\
+ \left( \frac{1}{\rho} \frac{\partial (\rho B_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial B_\rho}{\partial \phi} \right) \hat{k}
\]

(50)

3 Spherical Polar Coordinates

All the methods we applied in the previous section for cylindrical polar coordinates can be applied in the same way to spherical polar coordinates.

To begin, let us recall some basics about spherical polar coordinates (see Figure 6). From this figure it is apparent that

\[
x = r \sin \theta \cos \phi \tag{51}
\]

\[
y = r \sin \theta \sin \phi \tag{52}
\]

\[
z = r \cos \theta \tag{53}
\]

Note carefully that the two angles, \( \theta \) and \( \phi \), are intrinsically different. \( \theta \) is a polar angle, that measures inclination with respect to an axis. \( \phi \) is an azimuthal angle, that measures a rotation about an axis.

3.1 Position Vector

The point \( P \) indicated by the black dot in Figure 6 has position vector

\[
\mathbf{r} = x \hat{i} + y \hat{j} + z \hat{k} = r \hat{r}
\]

(54)

This might look even stranger than (4), but you should now expect position vectors in curvilinear coordinates to have information contained within the (non-constant) unit vectors. Here \( \hat{r} = \hat{r}(\theta, \phi) \) contains all the direction information about \( \mathbf{r} \). That is, to
get to a point \( P \), you travel a distance \( r = |r| \) in the \( \hat{r} \) direction.

From (54) we can write down

\[
\hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}
\]  

(55)

Differentiating this with respect to \( \theta \) and \( \phi \) will lead us to unit vectors in the direction of increasing \( \theta \) and increasing \( \phi \) respectively:

\[
\frac{\partial \hat{r}}{\partial \theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} = \hat{\theta}
\]

(56)

\[
\frac{\partial \hat{r}}{\partial \phi} = -\sin \theta \sin \phi \hat{i} + \sin \theta \cos \phi \hat{j} + 0 \hat{k} = \sin \theta (-\sin \phi \hat{i} + \cos \phi \hat{j}) = \sin \theta \hat{\phi}
\]

(57)

You should convince yourself that \( \hat{\theta} \) and \( \hat{\phi} \) are indeed unit vectors and that (56) and (57) give directions that agree with what trigonometry would tell you from Figure 6.

### 3.2 Line Element

Looking at Figure 6 to reveal the displacement vectors resulting from increments \( dr, d\theta, \) and \( d\phi, \) or taking the differential \( dr \) of the position vector (54) and making use of (56)–(57) leads to the expression for the line element in spherical polar coordinates:

\[
dr = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}
\]

(58)

### 3.3 Surface Elements

With reference to Figure 6 we can write down the basic surface elements of the three faces that meet at the black dot \( (r, \theta, \phi) \):

\[
dS_{r} = r^{2} \sin \theta \, d\theta \, d\phi \, dr \hat{r}
\]

(59)

\[
dS_{\theta} = r \sin \theta \, d\phi \, dr \hat{\theta}
\]

(60)

\[
dS_{\phi} = r \sin \theta \, d\theta \, d\phi \hat{\phi}
\]

(61)

We will see these again in Sections 3.6 and 3.7 when we work out the divergence and curl in spherical polar coordinates.

### 3.4 Volume Element

The volume of the element shown in Figure 6 is the product of the three orthogonal edges, i.e.,

\[
dV = r^{2} \sin \theta \, dr \, d\theta \, d\phi
\]

(62)

![Figure 7: Volume element in spherical polars with surface elements marked for application of the Divergence Theorem](image)

3.5 Gradient Operator

We can find the gradient through the exact differential as for cylindrical polars:

\[
df = \frac{\partial f}{\partial r} \, dr + \frac{1}{r} \frac{\partial f}{\partial \theta} \, d\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \, d\phi
\]

(63)

\[
\nabla f \cdot \hat{r} = \frac{\partial f}{\partial r} \, dr + \frac{1}{r} \frac{\partial f}{\partial \theta} \, d\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \, d\phi
\]

(64)

\[
\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}
\]

(65)

Comparing (63) and (65) gives

\[
\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}
\]

(66)

3.6 Divergence Operator

We derived the form of the divergence in spherical polar coordinates in lecture by a method that mimics closely that employed in Section 2.6. Here we will outline the key steps which follow from the divergence theorem applied to the volume depicted in Figure 7.

Recall (26)

\[
\nabla \cdot \mathbf{B} = \sum_{i=1}^{6} \mathbf{B} \cdot \frac{dS_{i}}{dV}
\]

Retaining only the dependencies related to the face in question we have

\[
\mathbf{B} \cdot dS_{1} = -B_{r}(r, \phi) r \Delta \theta \, r \sin \theta \Delta \phi
\]

(67)
in which you should recognise the forms of the different surface elements given in (59)-(61) with minus signs in some places to ensure that all the \(dS\)'s point out of the volume. Summing these pairwise and letting the volume shrink to differential proportions yields

\[
\sum_{i=1}^{6} \mathbf{B} \cdot d\mathbf{S} = \frac{\partial}{\partial r} \frac{r^2 B_r}{dr} dr \sin \theta d\theta d\phi + \\
\frac{\partial}{\partial \theta} \frac{\sin \theta B_\theta}{d\theta} d\theta r dr d\phi + \\
\frac{\partial}{\partial \phi} B_\phi r dr d\theta \tag{74}
\]

Dividing this summation by \(dV = r^2 \sin \theta dr d\theta d\phi\) then produces the desired expression:

\[
\nabla \times \mathbf{B} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2 B_r}{dr} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta B_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} B_\phi \tag{75}
\]

### 3.7 Curl Operator

As in the calculation of \(\nabla \times \mathbf{B}\) in Section 2.7, we shall apply Stoke’s Theorem to path integrals around faces of the volume element (Figure 7) to find the components of \(\nabla \times \mathbf{B}\) in spherical polar coordinates. The three faces, area elements \(d\mathbf{S}\), and corresponding right-handed paths are shown in Figure 8.

#### 3.7.1 \(\hat{r}\) component

The bottom sketch in Figure 8 shows the face of the volume element at radial distance \(r\) and that therefore has its surface element directed in the \(\hat{r}\) direction. (We take it to be in the +\(\hat{r}\) direction here, which is opposite to what is sketched as \(d\mathbf{S}_1\) in Figure 7). Applying Stoke’s Theorem to this face gives

\[
\sum_{i=1}^{4} \mathbf{B} \cdot dr_i = B_\phi(\theta, \phi) r d\theta +
\]

#### 3.7.2 \(\hat{\theta}\) component

The top sketch in Figure 8 shows the face of the volume element at polar angle \(\theta\) and that therefore has its surface element directed in the \(\hat{\theta}\) direction. (We take it to be in the +\(\hat{\theta}\) direction here, which is opposite to what is sketched as \(d\mathbf{S}_3\) in Figure 7).
Applying Stoke’s Theorem to this face gives

\[ \sum_{i=1}^{4} \mathbf{B} \cdot d\mathbf{r}_i = B_\phi(r, \theta) r \sin \theta \Delta f + \]
\[ + B_r(r, \phi + \Delta \phi) \Delta r - B_\phi(r + \Delta r, \phi) (r + \Delta r) \sin \theta \Delta \phi - B_r(r, \phi) \Delta r \]
\[ = (\nabla \times \mathbf{B})_\phi r \sin \theta \Delta \phi \Delta r \] (78)

Dividing by \( r \sin \theta \Delta \phi \Delta r \), re-arranging, and letting \( \Delta \phi \to d\phi \) and \( \Delta r \to dr \) yields

\[ (\nabla \times \mathbf{B})_\phi = \left( \frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r B_\phi)}{\partial r} \right) \] (79)

### 3.7.3 \( \phi \) component

The middle sketch in Figure 8 shows the face of the volume element at azimuthal angle \( \phi \) and that therefore has its surface element directed in the \( \phi \) direction. (We take it to be in the \( +\phi \) direction here, which is opposite to what is sketched for the front face \( dS_5 \) in Figure 7). Applying Stoke’s Theorem to this face gives

\[ \sum_{i=1}^{4} \mathbf{B} \cdot d\mathbf{r}_i = B_r(r, \theta) \Delta \phi + B_\phi(r + \Delta r, \theta) \Delta r - B_\phi(r, \theta + \Delta \theta) \Delta r - B_r(r, \theta - \Delta \theta) \Delta \phi \]
\[ = (\nabla \times \mathbf{B})_{\phi} r \Delta \theta \Delta r \] (80)

Dividing by \( r \Delta \theta \Delta r \), re-arranging, and letting \( \Delta \theta \to d\theta \) and \( \Delta r \to dr \) yields

\[ (\nabla \times \mathbf{B})_{\phi} = \frac{1}{r} \left( \frac{\partial (r B_\phi)}{\partial \theta} - \frac{\partial B_r}{\partial \phi} \right) \] (81)

### 3.7.4 \( \nabla \times \mathbf{B} \)

Collecting the components of \( \nabla \times \mathbf{B} \) from (77), (79), and (81) gives our final result

\[ (\nabla \times \mathbf{B}) = \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta B_\phi)}{\partial \theta} - \frac{\partial B_\phi}{\partial \phi} \right) \hat{r} + \]
\[ + \left( \frac{1}{r \sin \theta} \frac{\partial B_r}{\partial \phi} - \frac{1}{r} \frac{\partial (r B_\phi)}{\partial r} \right) \hat{\theta} + \]
\[ + \frac{1}{r} \left( \frac{\partial (r B_\phi)}{\partial r} - \frac{\partial B_r}{\partial \theta} \right) \hat{\phi} \] (82)

### 4 General Orthogonal Coordinate Systems

The methodology used in this handout can be applied to other orthogonal curvilinear coordinate systems. You will also find this done in several books by pure mathematical manipulations. All start from the cornerstone of coordinate systems, namely a set of unit vectors \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \), and the corresponding scale factors \( h_1, h_2, h_3 \) that convert differentials in the coordinates \( (u_1, u_2, u_3) \) into vector differential line elements, i.e.,

\[ dr = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \] (83)

You might try to replicate the approach here for such a general system, and then compare your answer with the results in Riley et al., or Boas.

### 5 Non-orthogonal Coordinate Systems

It is possible to define coordinate systems in which the three base unit vectors are neither straight nor orthogonal. We will not study such systems here. It is possible to derive corresponding vector calculus expressions for such systems using the fundamental machinery we have developed in the course. This must be done carefully; the biggest complication is that dot products between such unit vectors are not zero.